AID TO ENGINEERING SOLUTION

JACKSON

ENGINEERING SOLUTION

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AID

TO

ENGINEERING SOLUTION

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PREFACE.

THIS work is intended to correspond with 'Aid to Survey Practice,' and to afford a succinct account of a simple general method of effecting engineering solutions, as well as to give a complete set of solutions useful to the engineer.

This branch of science being dependent on successive development, much that is old will necessarily be found here, though mostly in some modified form. The more strictly new portions are some solutions in Horizontal Girders, principally continuous, constituting an extension of formerly known results; and the solutions in Braced Piers.

The remaining new portions are detached and scattered throughout the book.

There is also some novelty in the general system of dealing with the whole subject; and some peculiarities requiring special mention.

In the solutions and resulting equations due to the author, or modified by him, quantities and terms are never

arbitrarily rejected as insignificant, tricks of mathematical concealment have been carefully shunned, and the deduced equations are invariably suited to any single set of units of measure. That is to say, if feet are adopted in length, and hundredweight in weight, feet and hundredweight must run through the whole equation; for miles and inches, or pounds and tons, or metres and centimetres, are never mixed by him in the same equation; although any units may be used.

As to computations, this book being already large, another volume will be required for illustrating numerical applications to structures of the results of solutions.

The critical tendency of the age, which ridicules the past fashion of adopting the shortest clipped formula that comes first to hand, now also imperatively demands proof positive, or the nearest possible approach to it. Hence the increasing need for a book of this sort.

L. D'A. J.

ACKNOWLEDGMENTS.

THE task of assigning every borrowed and modified solution to some originator and to some date is an impossible one, or would require a lifetime to be devoted to it alone.

The acknowledgments of the author of this book are chiefly due to two persons, or their works: namely, to Stoney for portions on Horizontal Girders, in part, up to a certain point; also to Moseley for portions on Arches, Abutments, and Walls, in brickwork and masonry.—Enlargements, developments, and modifications have been effected on the bases they laid down in these two branches; and the simpler earlier parts have been borrowed from them; these amount to about half of the Section on Horizontal Girders, and three-quarters of the Section on Brickwork and Masonry.

The next most important borrowing is the general method of Chauvenet for Curved Ribs and Metallic Arches, on which basis the Section on that subject has been developed.

The fullest acknowledgments are therefore made to these three writers and developers of these three important branches of the Science.

As for the smaller direct borrowings, the acknowledgments for the sources from which they have been borrowed are given accompanying them; some of these are transmitted borrowings of the third or fourth order.

As a special exception may be noticed Cunningham's portions on Partial Loading of Horizontal Girders, and his two tables of quantities attached to the Theory of Three Moments. The Section on Continuous Girders was otherwise complete, and in-

volved a few corresponding alterations, after this recent addition. These portions, though small, are valuable, as well as directly useful. Correspondingly also, the recent solution of Allievi for a doubly convergent pier, and his reduction with fixed bars, have been incorporated with the results of the author's investigations, which had been conducted on the same principles several years before.

There yet remains the whole of the basic or simpler principles of Constructive Mechanics, which, however much altered, are certainly due to some one.

While much of our Applied Mechanics has been due to the French, and not very frankly taken by Whewell, Moseley, and others, there was yet an original English nucleus before the period of Whewell's writings.

This nucleus, surviving until now in the form of the pedagogic mechanics of our schools and universities, was not always viewed with contempt. At present, sonorously inculcated by ecclesiastics incapable of applying it, and ignorant of its applications, it is a degraded moribund curriculum ; formerly, utilised to the full by those practically cognisant of its endless applications, it was the expression of living science ; though crude, it was valuable and effective in relation to the wants and conditions then existing.

The first English collection of these principles appears to have been made about the end of the last century by Thomas Jackson, LL.D., and afterwards embodied in a series of lectures by him, when Lecturer on Applied Mechanics at Glasgow. He, or some of his family for him, founded professorships or scholarships at Cambridge and other places, for the encouragement or this special science. Hence, for this portion of the subject, full acknowledgment is due to him, as the national founder of the science, for all basic principles not attributable to more modern sources.

L. D'A. J.

CONTENTS.

PART I.

GENERAL PRINCIPLES OF CONSTRUCTIVE MECHANICS.

CHAPTER I.

STRUCTURAL STRENGTH.

CHAPTER II.

LOAD AND STRESS.

TABLES OF LOAD.

CHAPTER III.

RESISTANCE AND STRAIN.

TABLES OF QUALITIES OF MATERIALS.

PART II.

ENGINEERING SOLUTIONS.

SECTION I.

HORIZONTAL GIRDERS AND CANTILEVERS.

SECTION II.

CURVED RIBS AND METALLIC ARCHES.

SECTION III.

MASONRY AND BRICKWORK.

Arches.

Abutments, piers and walls.

SECTION IV.

PIERS, SUPPORTS AND STANCHIONS.

SECTION V.

MISCELLANEOUS SOLUTIONS.

AID TO ENGINEERING SOLUTION.

The Publishers will be glad to receive corrections of any errors that may occur in the First Edition of this work.

parts, and harmonious arrangement of the whole, of any engineering structure. It is a special branch of the general science of mechanics, limited by the nature of the subjects of its application.

The importance and vastness of this science render it necessary that every available means, mode, and device, by which solutions and computations can be effected, shall be used in accordance with the requirements of the particular subject or case under consideration. Even then, the powers both of science and of observation fail to determine for us a multitude of questions, with regard to which information would be useful and valuable.

It is therefore needful to assume that those perusing or referring to this volume for information possess such advantages as are conferred by an ordinary education, comprising elementary mechanics and elementary calculus, without any advanced knowledge of either subject. Apart from this assumption it would be impossible to comprise in one volume the demonstrations even of the comparatively limited number of solutions now known; while the increased labour, resulting from neglecting the use of the calculus and practising *détours*, and from using multitudes of additional coefficients and symbols, would alone constitute a serious objection, even if more space were available.

Under the assumed conditions the task of collocating engineering solutions and explaining the principles on which they are effected becomes much simplified, and the results are more easy to those reading or making reference.

Structural Strength.—The strength of a structure is said to be measured by that of its weakest part, both according to popular maxim and ordinary custom. This is therefore a fully recognised acknowledgment that the strength of a structure when taken as a whole cannot be arrived at by any direct process. An experienced engineer may, however, by inspection, judge correctly of its balance of strength.

Experiments on the strength of large complete structures have not yet been carried out, while any experience gained through accidental destruction of them, being obtained under conditions and circumstances not fully or accurately known, has thrown but little light on the subject.

The general stability of a complete structure, when treated as perfectly uniform and continuous, is generally investigated with reference to its weakest or least stable plane, and under the least favourable conditions, so that the above-

mentioned maxim repeats itself even applied to the whole.

Beyond this, it points to the necessity of discovering the weakest part of a structure, and this in many cases demands the investigation of the strength and stability of a large number of its parts.

Practically, therefore, the most important part of utilised constructive mechanics resolves itself into a series of mathematical determinations of the strength and stability of a large number of parts of structures of many sorts, dependent on equations of stress and strain, or of external force and internal resistance.

Sometimes these stresses and strains may be dealt with in groups, and sometimes otherwise, but under all circumstances the relation between stress and strain at any and every portion of a structure should be determined, either with accuracy or within some reasonable degree of approximation, in accordance with the requirements of the case and the powers of science to deal with it.

Treating this as the most important of the basic principles on which a work on constructive mechanics should be arranged, it becomes needful to enter first into the general principles involved in stresses, secondly into those affecting strains, and thirdly into the solutions affected by both of them in typical structures, or parts of structures of various kinds.

Stress and Strain.—*Stress* may be due to any direct external force, power, load, pressure, wind, change of temperature, or it may be an indirect transmitted action, resulting from any of such causes; thus with regard to any point, section, plane, or part of a structure, the stress on that part is the *local action* of external force.

(This explanation of the term stress requires one excep-

tion; there are also stresses due to the weight of the portion itself of the structure under consideration.)

Strain is the amount of resistance afforded by the cohesion of material at any point, section, plane, or part of a structure, and is a *local reaction* induced by a stress.

When there is equilibrium at the part under consideration, the strain is equal to the stress.

(This distinction between the terms stress and strain is not always adhered to; the two are sometimes treated as synonymous; occasionally also the term strain is used to represent the stretch, elongation, or alteration of form caused by a stress not perfectly counteracted; this latter custom is faulty, but fortunately it is not a very prevalent habit thus to confound cause and effect.)

The relation between stress and strain is identical with that existing between direct load and direct resistance.

As stress consists in relatively external force of any sort, either directly or indirectly transmitted as reactions from neighbouring parts, it follows that on leaving the consideration of stress and strain on one part of a structure, and going on to the consideration of stress and strain at the next neighbouring part, some of the stresses of the former part may represent *in amount* some of the strains of the latter part.

Thus the forces that are external with reference to one part of a structure may not be so with regard to a neighbouring part, but are then evidence of internal resistance or of transmission, as the case may be.

Each part of a structure may thus require perfectly independent consideration.

There are four basic principles employed in engineering solutions.

1st. Direct static reduction of stress, inducing strain.

2nd. Effect on the structure, whole or part, of any small displacement of the point of action of stress.

3rd. Deformation of structure due to supposed excessive stress, and immobility under permissible stresses of the same nature.

4th. Elastic deformation of material under stress.

The Equation of Stress and Strain.—This may take either of the two following *representative* forms, the first with direct forces and resistances, the second with moments of transmitted stress and summed resistances or strains:

$$\Sigma F = \Sigma R; \qquad \Sigma Ff = \Sigma Rr;$$

where ΣF is the sum of any number of stresses of the type F, or a total stress F; ΣR is the sum of any number of strains of the type R, or a total strain; ΣFf is the sum of the moments of corresponding stresses, ΣRr is the sum of the moments of any number of corresponding strains; and where the strain must be below certain values, *thus*: For simple equilibrium, the strain R must not exceed the *ultimate* strength of the material; for stability, the strain R must not exceed the *proof* strength of the material, that is, the strain it will bear without appreciable loss of cohesive power.

It is hence needful to have tables of ultimate resistance and of proof strength; or, failing the latter, coefficients of reduction whereby they may be fairly obtained.

(This part of the subject will be again dealt with when treating of strains; in the meantime it may be noticed that if a theoretically correct proof strain be not arrived at, but some approximate proof strain be practically applied to the material, which will induce a permanent set, then any less strain subsequently applied will cause no further effect.)

These conditions hold either for strength or stability at every section and in every plane of each part of the structure under consideration, and are applied to such sections of such parts as might from simple examination appear to be liable to deficiency in either respect.

The above representative equations may be actually very complicated expressions, involving symbolic dimensions, terms, and factors dependent on the form and disposition of the structure and its loading, and the nature of the resistances. The solution of such equations has for its object the determination of some explicit relation between the forces and resistances, or the value of some one or more of its terms or of symbolic dimensions of parts of the structure, that will hold under the equations of condition.

The basic principle above mentioned, which consists in mere resolution of stress and its direct application in inducing strain, is termed statical reduction.

Alteration of Form.—But when the above are the sole equations possible, there is then an assumption that they remain unmodified, whatever the stresses and strains may be, from their lowest value up to the values either at the verge of rupture or at the verge of deterioration of material; in other words, they hold for rigid, or nearly rigid, structures, and for materials whose powers are unimpaired, regardless of stiffness and elasticity.

If, on the contrary, the structure or the material is either flexible or elastic, an alteration of form may accompany a variation of amount in the stress or load, and besides, the strain and resistance may correspondingly vary in form of function.

Under such circumstances we may assign certain practically convenient values to the displacement caused by an unvarying stress, or we may alternatively adopt varying

stress, and at two definite limits have a pair of representative equations for the stresses, as load-functions :

$$\Sigma F_1 = \Sigma R_1 ; \qquad \Sigma F_1 f_1 = \Sigma R_1 r_1 ;$$
$$\Sigma F_2 = \Sigma R_2 ; \qquad \Sigma F_2 f_2 = \Sigma R_2 r_2 ;$$

or, if preferred, we may take three or more assigned values, and use as many pairs of representative equations. From these the required relations or values of quantities or terms may be determined corresponding to the assigned conditions.

Further, we may observe that under variation of stress, certain terms or quantities expressed in the general equations of stress and strain may remain constant, while others may be variable; in that case the general equation may take the form of a differential equation, from which by integration and successive integration other equations may be derived, each of which will hold good independently under prescribed limits.

From such equations the law of variation of any particular quantity or term may be sometimes deduced in the form of an equation to some curve. Such results in combination with the original equations of condition may yield required values of terms and quantities under conditions of varying stress, or of altered form.

The principles involved in these equations partially correspond to those adopted in virtual velocities. When stresses are dealt with in groups and with reference to particular planes, a general expression for the group of stresses and for the group of strains, or for a set of values of any required quantity, may possibly be arrived at. Such general expressions render it possible to construct diagrams or draw curves to scale, from which the required values may be taken by scale at any point within the group.

The alteration of form due to inherent elasticity of material can only be treated in solutions when the elasticity is dealt with as *perfect*; this assumption is practically correct for most constructive materials, when the elastic strain is less than the proof strain.

Under this condition, if the modulus of elasticity of the material suited to the case be known, also the values of the applied stresses, the change of form effected on a body of known original form may be deduced in a comparatively limited number of cases; but these forms must be symmetrical in some respect, and the results will hold only with regard to certain planes or sections.

The equation of elasticity.—The form commonly used equates the total external stress or its moment with the sum of the internal elastic strains at both sides of some neutral plane, thus:

$$\Sigma(F)=\Sigma(E_1+E_2); \qquad \Sigma(Ff)=\Sigma(E_1e_1+E_2e_2);$$

where E_1, E_2 are elastic strains of different sorts that may be induced expressed in typical form. In some cases E_1, E_2 are of the same sort, the elastic resistances can then be more directly dealt with.

Beyond this, there is also the basic equation of stiffness, which is really merely the evaluation of temporary set (often elongation under tensile stress). In its simplest application, which occurs with a section under tensile stress, it is a direct proportion

$$S.\frac{l}{L}=\frac{F}{E};$$

where S is the sectional area strained by the force F,
L is the original length of the body,
l is the elongation of the body caused by F,
E is the elasticity of the material.

In other cases this principle is applied to such various forms as may occur, in order to deduce alteration of form under strain.

Hence with elastic bodies we have three basic equations: (1) of direct counteracting force; (2) of force moments; (3) of elasticity; but with rigid bodies we have only the first two.

The number of deduced equations holding at various points of a structure and under various conditions may be very large indeed, and they may take strangely complicated forms; yet their fundamental principles must be those of the basic equations mentioned before.

Work of forces.—When stresses and strains are estimated in work exerted and work impressed, the equations are still those of stress and strain, but are employed in the form of work-stresses and work-strains.

Such is the general type of engineering solutions of detached cases.

Before proceeding to deal with particular solutions, the subject of stresses and that of strains may be generally dealt with in a manner that will aid in simplifying those solutions, and will prevent much repetition.

CHAPTER II.

LOAD AND STRESS.

Load.—When we refer stresses back to the sources from which they are derived, we find these to be external loads, weights, and pressures; and frictions in the same way: if we refer strains to their sources, they are correspondingly the resolved parts of the resisting powers of material brought into reaction, or induced by the stresses. It is therefore necessary to enter generally both into the nature of loads on structures, and into that of resistances of materials.

The generic term *load* or *loading* applied to engineering structures, includes both direct weight per unit of surface, and direct pressure per unit of surface; the *total load* being the whole load on the whole surface.

A load, or a pressure on a structure or a part of a structure, may be of any one of the following kinds, or of several:

1. Permanent load: weight of superincumbent structure or adjacent structure; earth pressure; water pressure; weight of traffic of any sort; reactions; and friction.

2. Weather load: due to weight of snow; soakage and result of rain; pressure of wind; inundation.

3. Accidental load: due to rare or special occurrence; such as the weight or pressure of an engine off the rails,

overturned vehicles, collision, suspended scaffolding and workmen during painting or repair; exceptional holiday traffic; transport of heavy guns and water-pipes.

4. Moving load: shocks, and suddenly applied load; involving a certain amount of dynamic force.

5. Inherent weight of the portion of structure under consideration.

Actual load consists in weight or pressure acting on a surface, and is estimated in units compounded of weight and of surface, as pounds per square inch, tons per square foot, hundredweight per square foot, or with decimalised units, in talents or foot-weight per square foot, or per square tithe. The cases in which actual load acts merely at a point, as, for instance, the weight of a suspended bell, or like a stone under a crane or traveller, are comparatively few in structural mechanics.

Representative collected load may, however, be assumed to act as simple weight or pressure on an edge or on a straight line across a section, or at a single point, or a series of detached points; any such assumption must, however, preclude all possibility of introducing error into any subsequent deduction: this must be watched throughout.

Moving loads, being dynamic, are reduced to their equivalent static loads; that is, they are generally simply doubled, before being incorporated with static load; the resulting stresses will then not require separate coefficients of safety, and solutions involving them are thus much simplified.

The following is the proof of the correctness of the reduction, as far as regards safety.

The work effected by a load W moved suddenly through a space s, is evidently $= Ws$. But if that same load be gradually applied through the same space s,

it increases during application from 0 to W; also the load W in this latter case varies uniformly with the space, and is hence $=ws$, w being constant: the work then effected

$$= {}_0^W\!\!\int {}_0^s\!\!\int \partial W . \partial s = {}_0^s\!\!\int W \partial s = {}_0^s\!\!\int ws . \partial s = \tfrac{1}{2} ws^2 = \tfrac{1}{2} Ws.$$

Now the effect of any moving load will be necessarily intermediate between that of a sudden load and a dead load; hence, with high velocities, as express trains, if the moving load be treated as a sudden load, the strength of structure estimated on this basis will be rather in excess. With low velocities, as those of road-traffic, probably multiplying by $1\frac{1}{2}$ would also yield excess of strength. (See 'Report of Commission on the Application of Iron to Railway Structures.')

Inherent weight of any part of a structure under consideration requires distinct treatment in accordance with the conditions of the case. In some cases the inherent weight does not affect the problem; in others, where it does affect it, such weight cannot have exactly the same effect as an equivalent amount of external load, although it may approximate to it. In many cases it is sufficient to estimate it as equal to half the same weight in external load. Experiment has not yet led to sufficient definite conclusions with regard to such comparison. It is, however, evident that whatever the ratio adopted in any case, the combination may be treated as a load under the same factor of safety. The alternative of using two separate factors, and of not making allowance by ratio, is far less convenient, although frequently adopted.

Both moving loads and inherent weight can thus be reduced to representative static weight, load, or pressure;

the stresses resulting from them after such reduction can then be treated as belonging to the same type.

In designing structures whose inherent weight has to be taken into account, although unknown, some approximate weight must, in the first instance, be assumed. With bridges and structures following well-known types, tables or diagrams can be used for this purpose. Among such tables and diagrams are those of Mr. B. Baker, in 'Long Span Railway Bridges' (London, 1867), and in a large diagram 'Weight of Girders,' published separately.

Permanent Load.—The actual permanent load on a structure may be determined by weighing the materials it has to carry, or by merely using tables of average weights of such material; similar methods may be also adopted with the rolling stock and traffic. (See Tables of Load.)

When the permanent load consists not of dead weight, but of lateral or resolved pressures, or of frictions, these must be separately estimated; in many cases the representative forces can only be arrived at through special solution.

It is, however, imperative in all cases, whether simple or complicated, to assume that collected representative loads or stresses act at points least favourable to the strength or stability of the structure within practical limits; for instance, the load on a pier may be conceived not on its vertical axis of form, but vertically at some possible lateral deviation, possibly close to a side of it; also the thrust from a semi-arch may be assumed to act at the most unfavourable edge of the middle third of the depth of keystone, and so forth.

As in many cases the resolution of total load, when carried into detail, enables stresses on any part of the

structure to be determined, the whole of such solutions may be treated generally as stress-solutions, without segregation.

Reaction.—Forces of reaction, induced by loading of any sort, have to be met in strength and stability of structure in the same way as original load-forces, or any induced or resolved stresses.

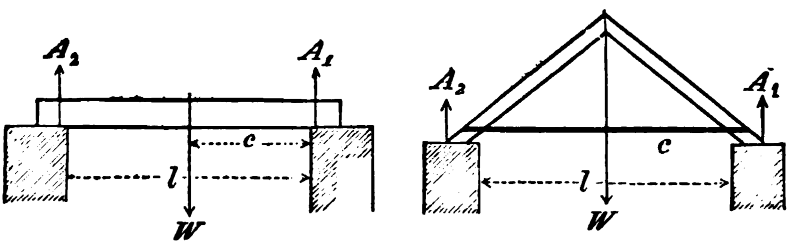


FIGURE 1.

Gross reactions may be deduced from the conditions of stability of a structure as a whole. Thus, with either a free beam or girder, or with a simple triangular roof-truss, as shown in the attached figures, we have with a concentrated gross load W the equations

$$\frac{A_1}{l-c} = \frac{A_2}{c} = \frac{W}{l}\text{; and } A_1 + A_2 = W;$$

with the roof-truss, if $c = \frac{1}{2}l$, we have then $W = 2A_1 = 2A_2$.

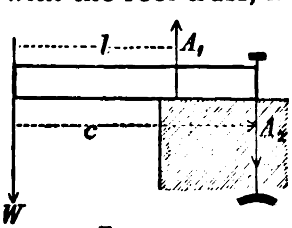


FIGURE 2.

When the beam is so inclined that the directions of A_1 and A_2 are inclined to that of the gross load, the values of A_1 and A_2 are obtained through the principle of parallelogram of resolved force.

When the reactions A_1 A_2 act in directly opposite

directions, as in the cantilever, a negative sign is introduced, and

$$\frac{A_1}{c}=\frac{A_2}{l}=\frac{W}{c-l}; \qquad W=A_2-A_1.$$

In the above cases a single concentrated load has been dealt with; but there are also cases when several concentrated loads apply at several points; also when a continuously distributed load is applied on a structure; in those cases the resultant of the various loads or the representative concentrated load has first to be obtained, and then dealt with as above, to obtain the corresponding reactions.

To obtain the position with regard to an origin of any such resultant or representative load, the principle of the centre of parallel forces is applied. For instance, with a free horizontal beam resting on two supports; let the loads w_1, w_2, w_3, w_4, &c., act at points situated at given distances

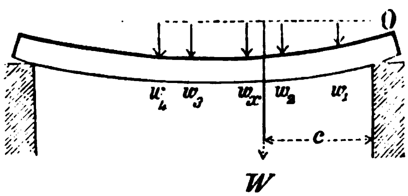


FIGURE 3.

c_1, c_2, c_3, c_4, &c., from O; let the resultant W act at a required distance c from O; then

$$c=\frac{c_1w_1+c_2w_2+c_3w_3+\&c.}{w_1+w_2+w_3+\&c.}; \qquad W=w_1+w_2+w_3+\&c.$$

Next, if on a similar beam the load be continuously distributed, let w be the load-intensity or load per unit of length of beam; then the load on an infinitely small length of the beam will be $w.\delta x$, where x is the abscissa measured from the origin O; and if W is the required

resultant, acting at the centre of gravity, then c is a particular value of the variable x, and

$$c = \frac{\int_0^x xw\,\partial x}{\int_0^x w\,\partial x}; \qquad W = \int_0^x w\,\partial x.$$

The value and the point of application of the single representative load being thus obtained, the origin can now be removed to the points of support or to the position of the resultant, and the required reactions can be estimated.

All such possible gross reactions, including any due to fixture, incastration, and continuity, should be estimated with regard to any structure, before proceeding into detailed stress.

Resolution of Load into Stress.

A load in its generic sense, which includes pressures and frictions, may be vertical, horizontal, or oblique, as regards absolute direction; similarly also, with reference to any portion of structure impressed, it may be longitudinal, transverse, or oblique, tangential, or rotary in action.

In many simple cases of applied load, the load is the stress. Thus, when the load acts direct and longitudinally with reference to any given portion of structure, the load on the section of material is stress; the stress is then resisted by the strain or internal resistance of the material, which is necessarily either tensile or compressive; these strains will be dealt with separately.

Next, when the applied load acts directly and transversely on a given portion of structure, as on a beam or on a girder, the resulting stresses are of two sorts, longitudinal or lengthway stress and transverse vertical coupled stresses.

Taking the simplest and ordinary case, when the beam

is horizontal, the load vertically downward, and the reactions act vertically upwards, the cross-section being laterally uniform, the whole action is confined to one representative plane; and the loads, forces, &c., may be expressed in units combined of weight and linear distance, as foot-pounds, inch-hundredweight, inch-tons, or foot-talents; the last being more convenient for calculation on account of the decimality and from forming part of a complete systematised series. (See Table of Compound Units.)

Stresses on a supported free beam or girder.

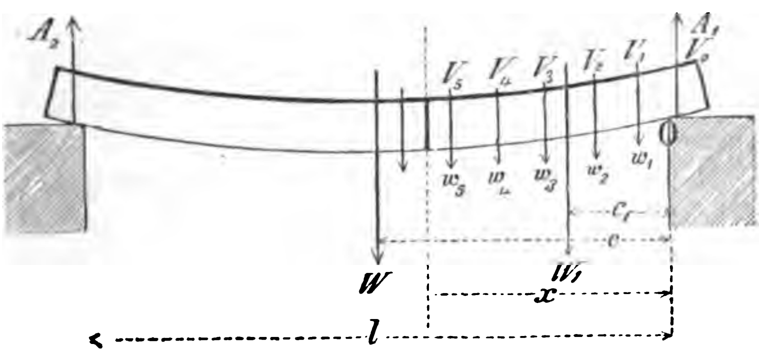


FIGURE 4.

The loading being all vertical, weight may be reduced to a single representative collected weight W acting at the centre of parallel forces; so that c is known. The values of A_1 and A_2, the reactions, are hence also known; *see* Reactions in a foregoing paragraph.

Let the origin O be at a point of support; now the vertical force at the point of support, where $x=0$, will evidently be—

$$V_m = A_1 = \frac{l-c}{l}.W.$$

The vertical force at any distance x from the origin that is less than c will be—

$$V_x = A_1 - W_1 = \frac{l-c}{l} W - W_1;$$

the components of W_1 lying to the right of the section under consideration. The same principle will hold for the value of V at any section.

Also with a continuous load,

$$V_x = A_1 - \int_0^x w\,\delta x = \frac{l=c}{l}.W - W_1.$$

The greatest value of V or V_m is evidently at the point of support, and the value of x when V=0 may be obtained through the above equations. If several sets of loads or portions of loads be applied, the total vertical force at any section will be the sum of the vertical forces obtained as due to each separate set at that section.

Next to estimate the horizontal forces, acting lengthways along the beam.

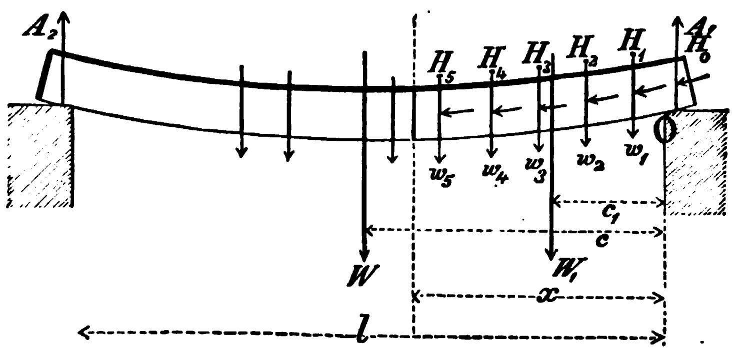


FIGURE 5.

Let the origin be at the same point of support A_1; then the horizontal force H at that point where x=0 will evidently be zero; and the values of H_1, H_2, H_3, &c. at the points where the component loads w_1, w_2, w_3, &c. act,

namely at distances x_1, x_2, x_3, &c., from the origin, will be thus :—

$$H_o = V_m \times 0 = 0$$
$$H_1 = V_m\, x_1$$
$$H_2 = V_m\, x_1 + V_{m-1}\, x_2$$
$$H_3 = V_m\, x_1 + V_{m-1}\, x_2 + V_{m-2}\, x_3$$
$$\&c. = \&c.$$

Examining this series, we may notice that the general term always includes the sum of the vertical forces up to any section ; and the ratio of the increment, whether finite or infinitesimal, may be therefore thus expressed,

$$\Delta\, H = V.\ \Delta\, x\,; \qquad \text{or } \partial H = V \partial x\,;$$

as the increment of $c_1 - x$ is equivalent to the increment of x; hence generally at any section

$$H_x = {}^x_0\Sigma\,.\ V_x\,.\ \Delta x\,;$$

and with a continuous load

$$H_x = {}^x_0\!\int V_x \partial x\,;$$

Or generally,

$$H_x = A_1 x - W_1\,(x - c_1) = \frac{l-c}{l}\ Wx - W_1\,(x - c_1).$$

From the above, it is evident that the values of H increase for sections further from the point of support ; and that H_m, the greatest value of H, will occur wherever

$$\frac{\partial H}{\partial x} = 0\,;$$

that is where $V = 0$; the position of which can be arrived at, as before explained.

This method is ascribed to Latham.

The results with loading of various sorts are collected in the following table :

General expressions for vertical stress V_x and for horizontal stress H_x at any point in a 'supported' beam or girder; in a length distant x from an origin O at one support.

Mode of loading	V_x	Position of V_m	H_x	Position of H_m	When the origin is at mid-span; H_x becomes
1. Uniform load throughout, w per unit of length on l	$w(\frac{1}{2}l-x)$	at origin; and at further support	$\frac{1}{2}wx(l-x)$;	at mid span	$\frac{1}{2}w(\frac{1}{4}l^2-x^2)$
2. Uniform load, w per unit of length up to c from O; stresses from origin to c	$w\left(c-\frac{1}{2}\cdot\frac{c^2}{l}-x\right)$	at origin	$w\left\{c^2-\frac{1}{2}\frac{c^2}{l}-\frac{1}{2}x\right\}$;	at $c-\frac{1}{2}\frac{c^2}{l}$	The corresponding reduction in other cases when the origin is changed, is analytically obtained by putting $\frac{1}{2}l-x$ for x, and $\frac{1}{2}l-c$ for c in the corresponding expressions, but it holds in the half-girders only, without further change of sign.
stresses beyond c.	$-\frac{1}{2}\frac{c^2}{l}$	. . .	$\frac{1}{2}\frac{wc^2}{l}(l-x)$		
3. Single load W at mid-span; in half of beam nearest origin	$\frac{1}{2}W$	at origin	$\frac{1}{2}Wx$	at mid-span	
in further half of beam .	$-\frac{1}{2}W$	at further support	$\frac{1}{2}W(l-x)$	at mid-span	
4. Single load W at distance c from O in portion c of span	$W.\frac{l-c}{l}$	anywhere	$W.x\frac{l-c}{l}$	at load	
in portion $l-c$ of span	$-W\frac{c}{l}$	anywhere	$Wc\frac{l-x}{l}$	at load	
5. Equal and opposite couples at the two ends W_1 and W_2 for load and reaction as from fixture	zero throughout the mid portion	. . .	W_1c, or $W_2(l-c)$ constant		

A moving load is to be represented by an enhanced value in the form of a steady load. *See* Load paragraph.

The maximum stresses produced by *passing* loads will be dealt with in a separate paragraph.

In any case of particular loading, a graphic representation, by lines or parabolic curves, of the vertical or of the horizontal forces, will enable a value of either to be scaled off for any point in the span, that is, for any section of it.

Stresses on a Cantilever.

The loading, being all vertical weight, may be reduced to a single representative collected weight W acting at the

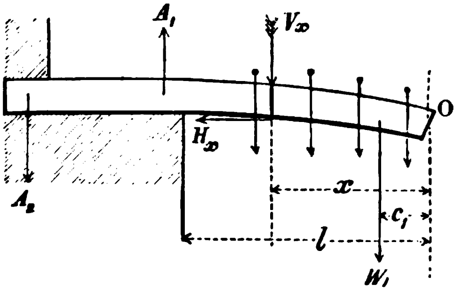


FIGURE 6.

centre of parallel force, so that c is known. The values of A_1 and A_2, the reactions, are hence also known ; *see* Reactions in a foregoing paragraph.

Let the origin be at the free end, and let the vertical stresses that act downward have a negative sign. Then the vertical force at O where $x=0$ will be zero ; and the vertical force at any point distant x from the origin will be downward $-V_x = W_1$; the components of W_1 lying to the right of the section under consideration.

The same principle will hold for any section up to the support, where $x=l$, and W_1 becomes W ; so that then $-V$ has its greatest value W.

Also with a continuous load:

$$-V = {}_0^x\!\int w.\partial x\,;$$

where w is the load per unit of length.

If several sets of loads or portions of loads be applied, the total vertical force at any section will be the sum of the vertical forces obtained as due for each separate set at that section.

The horizontal forces acting lengthways along the cantilever may be estimated in precisely the same way as those for a supported beam or girder, in the preceding paragraph, arriving at the two results:

with detached loads, $H_x = {}_0^x\Sigma\, V_x.\Delta x$;
with continuous loading, $H_x = {}_0^x\!\int Vx.\partial x$;
or generally, $-H_x = W_1(x - c_1)$.

The value of H evidently increases at sections nearer the point of support; and H_m the greatest value will occur at the point of support. At the free extremity $H = 0$; where also $V = 0$.

General expressions for vertical stress V_x and horizontal stress H_x at any point in a cantilever of length l, distant x from an origin O at its free extremity.

Mode of Loading	V_x	Position of V_m	H_x	Position of H_m	When the origin is at the point of support; H_x becomes
1. Uniform load throughout w per unit of length l	$-wx$	at l	$-\frac{1}{2}wx^2$	at l	$\frac{1}{2}w(l-x)^2$
2. Single load $= W$ at the extremity	$-W$		$-Wx$	at l	$w(l-x)$
3. Twofold loading, combining loading of Nos. 1 and 2	$-(W+wx)$	at l	$-(Wx+\frac{1}{2}wx^2)$	at l	$\{W(l-x)+\frac{1}{2}w(l-x)^2\}$

Stresses on a Fixed Beam or Girder.

The loading and the stresses mostly correspond to those on a free beam or girder already dealt with at pages 17–20; it is hence merely necessary to indicate the mode and the amount by which they differ.

The direct weight between supports W and its reactions A_1, A_2, being the same as in the former case, the effect of fixing the ends of the beam is to introduce two equal representative forces $-a_1$, $-a_2$ acting vertically downward, and their reactions a_1, a_2 acting vertically upward at the points of support.

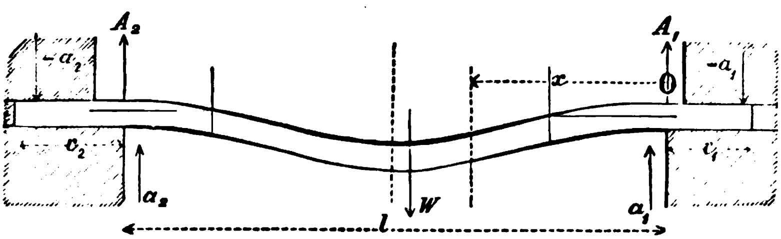


FIGURE 7.

Such a pair of reaction couples may be treated as an additional loading of a special sort, applied to a supported girder. Their effect on the vertical forces on the beam between supports is evidently nothing; hence the values of V_x tabulated for cases of supported beams will also hold for fixed beams.

Their effect on the horizontal forces, acting along the beam produced by direct weight, will evidently be to counteract them to some extent.

Hence it is necessary merely to estimate the horizontal force produced by the reaction couples independently of other conditions of load, and to subtract the result from the value of H_x already given for a free girder, in an otherwise

similar case. The values a_1+a_2 constituting the load, the horizontal force between supports is

$$h_x = a_1 c_1 = a_2 c_2 = a_1 x = a_2(l-x)$$

for every point or section distant x from the origin O, and is hence constant. See also case 5 in the Table of Stresses on a supported girder, at page 20.

The correct value of the horizontal stress for the fixed girder is hence $= H_x - h_x$, the values of both of which are now known; it will have a positive maximum value, when H_x is greatest; and it will have a negative maximum value, when $H_x=0$, being then simply $-h_x$, and this is the maximum that most frequently occurs in practical cases.

The curve of horizontal stresses for a fixed girder is hence identical with that for a free girder, as all the values vary merely by a constant; but graphically the ordinates will be measured from an axis parallel to the axis employed for the similar free girder.

A fixed girder is divisible into three parts, a central portion as a supported girder, and two cantilevers, one at each end; the neutral points, or points of contrary curvature occur wherever $H_x = h_x$.

General expressions for total horizontal stress at any point in a 'fixed' beam or girder, distant x from an origin O *at mid-span.*

Mode of Loading	$H_x - h_x$	Position of maximum	V_x
1. Uniform load throughout w, per unit of length l	$\frac{1}{24}wl^2 - \frac{1}{2}wx^2$	Explained above	as in similar free girders.
2. Single load $(-W)$ at the middle	$\frac{1}{8}Wl - \frac{1}{2}Wx$		
3 Twofold loading, combining loads (1) and (2)	$W(\frac{1}{8}l - \frac{1}{2}x) + w(\frac{1}{24}l^2 - \frac{1}{2}x^2)$		

Stresses on a Continuous Beam or Girder.

The resolution of load, reactions, and the reduction of the whole of the stresses in continuous beams cannot be *entirely* effected, apart from the consideration of the strains induced; that is to say, some of the stress-values are analytically dependent on the elasticity of the material, a subject that will be first dealt with separately.

It will, however, be convenient, while running through the subject of stresses, to deal with the distribution of those of a continuous girder in a representative form, apart from their valuation.

The general condition of a continuous girder of several spans is that each span between piers consists of a sup-

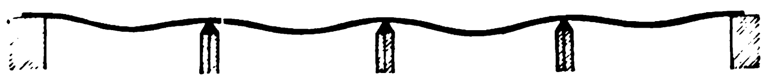


FIGURE 8.

ported girder resting on two cantilevers, while each end span between a pier and an abutment consists of a supported girder resting on a cantilever towards the pier and on the abutment itself.

Special conditions of loading may, however, cause certain of these constituent divisions to be so extended as to reduce others to evanescence.

Treating one of the central spans in comparison with a single detached span of supported girder, it is clear that the effect of continuity is to introduce increased reaction at the supports, induced by the load on contiguous spans, to augment the vertical force at the supports, also to increase the horizontal forces there.

Thus, the reaction at one support may be represented by $A_1 + a_1$; at the other by $A_2 + a_2$; and if $H_1 + h_1$, $H_2 + h_2$

be the augmented horizontal force at each corresponding

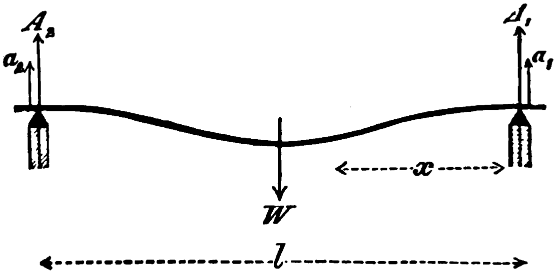


FIGURE 9.

support, we have by taking moments round each support in turn,

$$H_1+h_1=H_2+h_2+a_2 l; \qquad H_2+h_2=H_1+h_1+a_1 l.$$

Also with the increased vertical forces, we have evidently,

$$V_1+v_1=A_1+a_1; \;\; V_2+v_2=-(A_2+a_2);$$

and generally

$$V+v=a_1+V=-a_2+V.$$

Besides the general relation between the vertical and horizontal forces, proved in page 19, still holds after their augmentation:

$$\Delta(H+h)=(V+v).\Delta x; \text{ or } \delta(H+h)=(V+v)\,\delta x.$$

Reverting to the augmented horizontal force generally at any section at x, we have:

$$H+h=H_1+h_1+a_1 x+H;$$

also as

$$(H+h)l-(H_2+h_2)x=(H_1+h_1)(l-x)+lH;$$

$$\therefore\ (H+h)=H_1+h_1+\frac{x}{l}\{H_2+h_2-H_1-h_1\}+H,$$

or

$$h=H_1+h_1+\frac{x}{l}\{H_2+h_2-H_1-h_1\}.$$

When we proceed to deal with a complete series of spans of a continuous girder, there is also the equality between the sum of the total loads and the sum of the total reactions, to aid in the evaluation of the reactions.

There are three basic principles applying in continuous girders generally.

First.—If in any span w=load-intensity per unit of length of span ; l=span ;

H_1, H_2 the horizontal forces or bending moments at each of the supports ;

H_x the horizontal force at any abscissa x from the support H_1 ; then :

$$H_x=\frac{H_1(l-x)+H_2x}{l}+\tfrac{1}{2}wx(l-x).$$

Second.—The sum of any such forces, h_x', h_x'', h_x''', *&c.*, due to separate loads, is equal to H_x, the force due to the total loads on the single span.

Last.—Any two contiguous spans of a continuous girder may be treated at one time by the theorem of Three Moments, which will be applied later in Part II.

Continuous and symmetric loading on the spans, which often occurs, aids in the reduction of the stresses ; but their evaluation depends on the properties of the elastic curve, through the strains induced, and will hence be treated among the solutions.

Stresses on a Bowstring Girder.

An ordinary bowstring girder takes the general form of a circular segment, the rise of which seldom exceeds one-eighth of the span ; and thus approximates to a parabolic segment, which is the true equilibrated form, relieving the string from all forces other than horizontal forces ; and

placing any length of bow from the vertex downwards on either side in true equilibrium under a uniform load. Through this the stresses on the bow may be determined.

(The equilibrated parabola, both erect and inverted, is treated in Elementary Principles of Mechanics, in connection both with arches, curved ribs, and suspension chains, and is hence a familiar elementary subject.)

If w be the intensity of load per unit of span, d=rise or mid-depth of girder, l=the span; there is an equilibrated couple acting at mid-span, having d for their leverage, which may be represented by H and h, horizontal forces. But h is

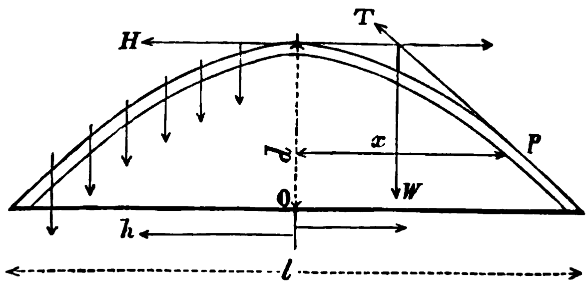


FIGURE 10.

actually the value H_m in a supported girder already determined$=\frac{1}{8}wl^2$; hence $H.d=\frac{1}{8}wl^2$, and H is thus known.

The thrust T at any point p, whose abscissa from a vertical axis at mid-span is x, can be determined from the equilibrium of H, T, and W the weight and load on the segment from the vertex down to p; whence

$$T=(H^2+W^2)^{\frac{1}{2}}=(H^2+w^2x^2)^{\frac{1}{2}},$$

and the greatest thrust T_m occurring at a support

$$=(H^2+\tfrac{1}{4}w^2l^2)^{\frac{1}{2}}=\frac{wl^2}{8d}\left(1+\frac{16d^2}{l^2}\right)^{\frac{1}{2}}=\frac{wl}{8d}\sqrt{l^2+16d^2}.$$

Hence as $\frac{d}{l}$ practically does not exceed $\frac{1}{8}$, T_m cannot exceed $\frac{1}{2}H.\sqrt{5}$.

Also the vertical force acting on the web, supposing it to be continuous, is w per unit of length along the span.

And the horizontal force acting as constant through the string is

$$H=\frac{wl^2}{8d}.$$

A similar treatment of stresses may be used with detached loads when their distribution is uniform.

A bowstring girder may have a continuous web, a braced web, or no web at all; in the latter case the vertical strains are all thrown on the bow. When, besides that, the string is omitted or dispensed with, the thrust is entirely met by abutments, and the bowstring girder is then transformed into a Curved Rib.

Extreme Stresses on Girders from a Passing Load.

The ordinary stresses already given are due to certain loads, either fixed or moving, of known position, their abscissæ along the span being given; the greatest values of such stresses that occur anywhere in the span are hence also known. But it becomes also of interest to learn the *extreme* values of the stresses that can be produced at each section when a load passes through all possible positions in transit; for the reason that all of them cannot in *all* cases be those produced by a load completely covering the span or spans, although that is frequently the case with stresses of some kinds.

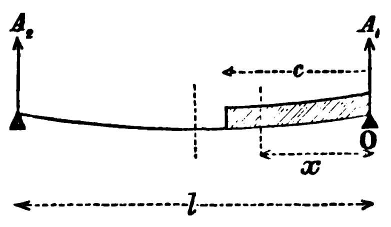


FIGURE 11.

First, with a supported girder under a concentrated passing load, the extreme values of the horizontal and

vertical forces, H_x and V_x, at any section whose abscissa is x, which we will symbolise by H_{ex} and V_{ex} will evidently occur when the passing load is at that section.

Secondly, with a supported girder under a uniform passing load that may cover the whole span : H_{ex} will occur simultaneously at every section when the load covers the whole span, but V_{ex} does not.

In fact, if we consider the value of V_x, ordinary vertical stress, for the section at x in the figure, we shall find that when the passing load extends to a distance c that is greater than x, the value of V_x is thereby diminished, and will continue diminishing as c increases. As we must hence look for increased values of V_x in the unloaded portion of the span, let us now suppose the load to cover the part $l-c$, leaving the part c unloaded, in which is the section whose abscissa is x (see figure 12). Then V_x is constant throughout c, and is there generally

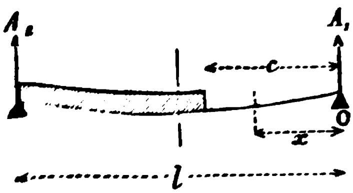


FIGURE 12.

$$V_x = A_1 = \frac{w'(l-c)^2}{2l};$$

where $-w'$ is the passing load-intensity per unit of length. Provided that the loaded part $l-c$ is greater than half the span, the extreme value V_{ex} will occur when $x=c$; that is

$$V_{ex} = \frac{w'(l-x)^2}{2l},$$

in the half span nearest to A_1: and this is, as before mentioned, greater than

$$\frac{w'(l^2-2x)}{2l},$$

the value occurring when $c=0$.

Corresponding also in the half-span nearest to A_2, the extreme value of

$$V_{ex} = -\frac{w'.x^2}{2l}:$$

and the sign changes when the load passes the section under consideration.

The maxima among the extreme values of V_{ex} will be when $x=0$, and $x=l$, that is, at the supports; and the single minimum among the extreme values is when $x=\frac{1}{2}l$, or at mid-span.

Thirdly, a supported girder under a steady uniform load $-w$, and a passing uniform load $-w'$ per unit of length.

In this case we can combine the results due to the separate cases:

$$\text{from } A_1 \text{ to mid-span, } V_{ex} = w(\tfrac{1}{2}l-x) + w'\frac{l-x}{2l};$$

$$\text{from } A_2 \text{ to mid-span, } V_{ex} = w(\tfrac{1}{2}l-x) - w'\frac{x^2}{2l};$$

Whence the maximum value of V_{ex} occurs at either support, and is $V_{ex} = \pm(w+w')\frac{1}{2}l$.

Also generally $H_{ex} = \frac{1}{2}x(l-x)(w+w')$; and the maximum value of H_{ex} occurs at mid-span, being there

$$H_{ex} = \tfrac{1}{8}l^2(w+w').$$

Fourth, a bowstring girder under uniform steady load w, and uniform passing load w' per unit of length of string, H_{ex} will occur simultaneously at every section when the passing load covers the whole span; but V_{ex} will occur in an unloaded part c, when the passing load covers a part $l-c$ that is greater than half the span.

The above instances show that a partial passing load

augments the vertical stresses in the unloaded portion of a span; the same principle also holds with continuous girders; the effect is to require additional strength of web to meet such stresses.

This subject has been ably treated in many cases by Cunningham in his 'Applied Mechanics;' his book was consulted during the revise of this paragraph.

Stresses on Braced or Framed Structures.

When a braced or a framed structure takes the general form of a closed figure, and consists of upper members, lower members and intermediate bracing, the stresses on the braces and on the booms of the upper and lower members may be arrived at on the following very simple general principle, which holds either for braced girders, bridge trusses, or roof-trusses.

(1.) The values of the horizontal stresses H_x, due to the general load, are represented by the ordinates of a curve drawn with reference to an axis of abscissæ whose length is the span; the values of V_x, the vertical stresses, are correspondingly represented by a curve.

(The general method of obtaining H_x and V_x for parallel girders, and of obtaining thrust T_x in bowstring girders has been already treated; the thrusts T_x in other cases are arrived at through simple resolution of force or of general load.)

The values of V_x and of H_x can be then taken out for a series of sections at all points of bracing in the upper and lower members, and resolved along the respective braces.

The inherent weight of any portions, braces or booms, that affect other braces or booms will be added to the

general stress or each brace or boom; thus obtaining the total stress on each brace or boom direct.

Some of these stresses may be contrary in sign in some cases, as is evident from the consideration of the values of V_{ex} in the foregoing paragraph.

This method, generally termed the 'method of sections,' is applicable also when the web is continuous.

(2.) When a braced girder or a truss consists of a *small number* of parts or pieces, the 'method of direct resolution of general load' may be preferable from greater convenience.

This process may be carried out in several ways, familiar in elementary mechanics, the principal of which are the two following; the first being usually applied to roof-trusses.

i. The load consisting of a general load over the whole frame, its reactions at supports, and the inherent weight of parts; the general load is first distributed at the middle points of each bar or boom on which it rests, and these distributed loads are then resolved into parallel components at the two ends of each bar or boom; the summation of these gives a series of loads at the joints.

Next, the inherent weight of any brace affecting the stress, or any revolved stress resulting from it, is added to or subtracted from the joint-stresses due to the general load; and the resulting stresses are resolved by parallelogram of force in the directions of the braces.

Should any additional normal or accidental load occur, it must be suitably resolved and compounded with these resulting stresses. In some cases there will be unstrained bars, that is the stress along these will be zero.

Finally, the reactions are compounded with the resulting stresses at the points of support.

This process may be verified by equating the sum of

the joint-stresses with the sum of the general, partial, and resolved normal loads introduced; each set being taken separately from the point of introduction.

It must be noticed that this method applies to frames that are loaded only at the joints, as trusses receiving load through purlins; should any bar be loaded transversely at any intermediate point, it becomes a beam requiring independent consideration on that account.

ii. With bridge-trusses, this last method applies until arriving at the joint-stresses; after that, the partial reactions of each of these joint-stresses, acting at the points of support, have to be separately determined; next the stress on each separate bar due to each of these separate partial reactions has to be determined by resolution; the sum of such stresses on each separate bar is the resultant stress on each bar, using algebraic signs to distinguish contrary directions of forces.

This process, whether i. or ii., becomes exceedingly tedious and laborious when the bracing consists of a *large number* of pieces; and is hence not recommended for such purposes. With roof-trusses of many parts, process i. may be adopted until the joint-stresses are obtained; after this their resolution through successive parallelograms of force may be effected graphically in a stress-diagram, thus avoiding both the tediousness and the uncertainty of numerical computation.

With bridge-trusses of many parts, process ii., said to be due to Latham, is fully exemplified by Rankine, in pages 550 and 559 of his work, where the length of example given serves as a most useful warning. It is preferable, after obtaining the joint-stresses, to make a stress-diagram, and obtain graphically the values of the stresses on the bars.

But even the graphic reduction is not well suited to brace-stresses under passing loads, although it applies well to boom-stresses and to a fixed load.

The graphic reduction of joint-stresses to bar stresses, either for process i. or ii., is generally familiar, and exemplified in so many elementary works that further exemplification is needless. The 'Method of Sections' being of general application is far preferable to either, excepting when the number of bars and booms is few.

Stresses on Curved Ribs.

The development of the curved rib from the simple bowstring girder may be briefly expressed.

If a bowstring girder be entirely destitute of web or bracing, the tie-rod or chord-member, having merely to resist tension, may evidently be also dispensed with, when the ends of the bow abut against any firm bearing, as a masonry skewback. The abutments then not only support the bow, but retain it against the extension, that would otherwise result from loading. The bow thus becomes a Curved Rib.

The rib may then resolve itself into two distinct members or flanges, an upper and a lower, connected by web or bracing.

Such curved or elastic ribs may take several forms, dependent on variation in the depth of section, in the breadth of it, or general variation throughout. The simplest form of curved rib is one of uniform section, after which there is the rib of uniform depth, with variation in the width of section, also the rib having only two sorts of section, the greatest portion in the middle being uniform, while two comparatively short pieces near the two ends are of increased section.

Ribs of these sorts, when supporting horizontal roadways, may have intervening material or struts for that purpose, but those are then mere loading, as far as the rib is affected. But when the depth of the rib varies, and intermediate bracing or spandril filling between the members is employed, another type of rib is entered on, and the whole of the strains and stresses are affected.

The curvature of a rib is generally either circular or parabolic, sometimes theoretically catenarian; the ends of it may be either simply held against extension at the supports, in which case it is termed a free curved rib, or they may be positively secured or fixed, a mode that adds to its stiffness and is most commonly adopted.

All curved ribs will evidently be liable to deflect under loading, and will also exert thrust. The determination of the stresses produced by a load can only be effected through consideration of the strains induced and of the properties of the elastic curve. Though it is premature to attempt it until the latter is done, it is yet possible to indicate the representative stresses produced by a load before attempting their evaluation.

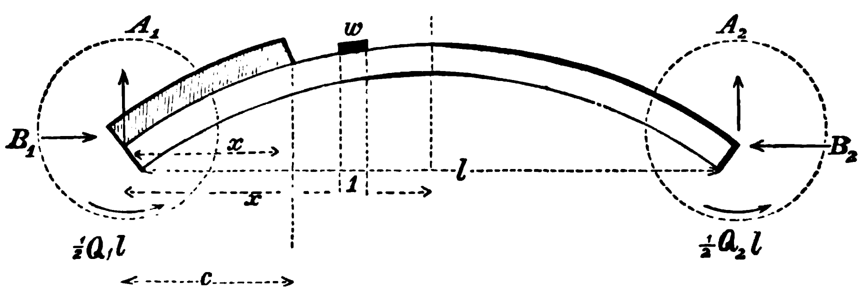


FIGURE 13.

Taking a fixed curved rib of uniform section and of circular curvature under an equally distributed load.

Let l = the length of span between bearings;

w = load per unit of horizontal length of l;
A_1, A_2 = the vertical reactions at either abutment;
B_1, B_2 = the horizontal reactions at either abutment;
$\frac{1}{2}Q_1l$, $\frac{1}{2}Q_2l$ = the rotary moments exerted at the fixed points, due to fixture.

Then with an equally distributed load the corresponding reactions &c. at the two abutments will be equal; and $A_1 = A_2 = \frac{1}{2}\ wl$.

Although fixture demands the introduction of the rotary moments, it also affords the condition that the inclination of the curve at the fixed supports remains invariable under any loading. On the other hand, if the rib were technically free, the curvature at the free supports would vary under different loads, although the rotary moments $\frac{1}{2}Q_1l$, $\frac{1}{2}Q_2l$ would vanish, or = zero.

Taking any transverse rib-section normal to the curvature, let x and y be the rectangular co-ordinates of its middle point from an origin at the fixed support A_1; the external forces acting on this section may be represented by a single force F acting at any point in the section, which need not be on the neutral axis, or neutral plane of the rib. This force F may be then resolved into two forces, one $F_1 = F$ but acting at the neutral axis at C, the other a couple whose moment H tends to cause rotary motion around C. Next, F_1 may be resolved into two forces acting at C, one normal N, the other tangential T.

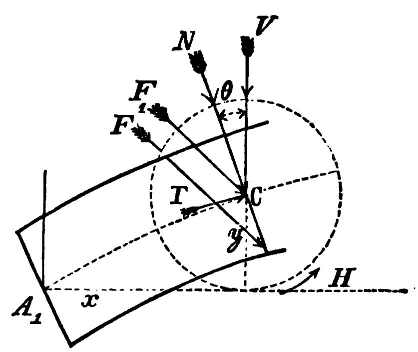


FIGURE 14.

Hence the whole external force acting on the section may be represented by H, N, and T stresses inducing

flexure, shearing, and thrust respectively: that will vary at every transverse section in the rib.

From the solution of the case, by equating the moment of sectional resistance with the moment of external force, the values of A_1 B_1 and $\frac{1}{2}Q_1 l$ can be determined, also the values of $H=M$, for a series of sections corresponding to equal distances along the span. With an equally distributed load the general expression for H at any section whose abscissa is x, and ordinate is y, is

$$\tfrac{1}{2}wx^2 - A_1 x + B_1 y + \tfrac{1}{2}Q_1 l.$$

Also to find N, and T; if θ be the inclination of the normal section to verticality, and V represent the vertical force acting at C on the neutral axis of the section; there is then

$$V = A_1 - wx\,;\quad N = B_1 \cos\theta + V\sin\theta\,;\quad T = B_1 \sin\theta + V\cos\theta.$$

Thus, at a series of sections throughout the half-span the values of the stresses can be determined; those in the other half-span will correspond.

Secondly, let the same curved rib be merely partially loaded to a horizontal distance c from the origin A_1; see figure 13.

In this case the moment of external force about any section, within the length $l-c$, will be when $x>c$:

$$H = wcx - \tfrac{1}{2}wc^2 - A_1 x + B_1 y + \tfrac{1}{2}Q_1 l\,;$$

and $V = A_1 - wc$.

In the loaded part where $x<c$; $V = A_1 - wx$.

Also for use in the other half-span, we should have the reactions at the right abutment

$$A_2 = wx - A_1\,;\quad B_2 = B_1\,;$$
$$\tfrac{1}{2}Q_2 l = -\tfrac{1}{2}Q_1 l + A_1 l - wc(l - \tfrac{1}{2}c)\,;$$

which would be used as required.

The above cases are those of a fixed rib; with a free rib, $\frac{1}{2}Q_1 l=\frac{1}{2}Q_2 l=0$. There are also two other conditions possible: one that the ends are pivoted, not fixed absolutely; the other that the abutments yield slightly, and the span therefore enlarges a little.

Further consideration of these stresses apart from the strains in a curved rib would be of little advantage.

The Arch.

Development of Stress-Conditions.—When treating of stresses hitherto, a course of development of structures, or rather of stress-conditions in connexion with them, has been followed, and will also be pursued onwards to the end of this chapter.

(It is not, however, presumed that the historic development of the structures themselves has followed this order; on the contrary, that has necessarily followed the chronological or successive employment of materials of different sorts, under various improved forms and modes of manufacture.)

When a continuous curved rib, whether as a concrete monolith, or as an elastic metallic rib, has its continuity destroyed and is made to consist of a number of pieces, such as blocks in the one case, or panels in the other, these pieces when put together in the original form, and artificially joined by cement or by bolts, form a true arch. The voussoirs of the arch are the blocks or the panels, either solid or hollow from piercing; the divisions or joints may be either radiated toward the centre of curvature of the arch, or may be vertical; but in any case the principles of the arch have been introduced; and these will not be materially affected by a further sub-division of the arch into courses of concentric rings, as is usual in brick arches, whether through-bonding be adopted or otherwise.

As regards stress, the principal result of the destruction of perfect continuity is that the pressure-curve is broken. Whether the joints are weaker or even stronger than the blocks or panels, they are not identically the same as regards resisting power.

Admitting the existence of a broken pressure-curve of some sort, it will follow one law in the block or panel, and another in the joint. Presuming that the joints are very narrow, the vertical deviation of the broken curve within each joint may be comparatively small; but even if granted to be zero, when we treat portions of the arch from joint to joint for determination of the points on the pressure-curve, the dissemination of the pressure through each block or panel remains unknown. So that, even under that hypothesis, static resolution merely can give points on the pressure-curve at any separate joints.

It may roughly be assumed that in the blocks the pressure will follow the line of least resistance between the points at the joints, that is, straight lines; but there is not sufficient warrant for the assumption.

As regards stress, also, the next point requiring practical consideration is that the materials of which arches are generally composed, such as stone, brick, and cast iron, are capable of but small tensile resistance; they are treated as rigid or comparatively inelastic, so that the equation of elastic deformation cannot be applied to the structures made of them. The general stresses are hence entirely dependent analytically on the broken pressure-curve before mentioned.

Supposing the broken pressure determined (a matter that will be afterwards dealt with), the details of pressure in the blocks or panels have also to be considered according to some assumed mode of dissemination. Small equal blocks or wedges of stone or bricks may be treated as

homogeneous solids; but when the bolted metallic panels of a cast-iron arch have to be dealt with, which are generally pierced, of large size and varying in form from the crown to the haunch, the analytical treatment is necessarily different.

Having thus generally indicated the stress-development ensuing in the formation of the arch, its variety and accompanying detail can be entered into.

The Stability of Blocks.—As either the typical arch or the typical abutment, pier, or wall may be structures of solid or hollow blocks, the general conditions of stress relating to blocks will be given before entering into further detail.

These conditions are three. 1st. The stresses tending to overturn a block or the whole of the blocks. 2nd. Those tending to cause one block to slide on another. 3rd. Those tending to crush the block, or produce subsidence in its foundations or supports.

1st. *Overturning.*—Given a series of blocks resting on each other, and acted on by any external loads and thrusts, we may by successive static resolution obtain the resultant stress acting at each joint both in magnitude and direction; and the intersection of each of these directions with the directions of the joints themselves will thus give a series of points. These points are, strictly speaking, points on a broken curve of equilibrium, both of pressure and of resistance. There is both stress and strain beyond this broken curve, which, as variously termed, is a line of resultants or a line of extreme stress and strain.

It is evident that if this broken curve fall outside of any joint at the produced direction of that joint, equilibrium cannot exist; but if it fall on the joint, equilibrium does exist. Thus, in figure 15, T_1, T_2 fall on their joints, and T_3 beyond its joint.

To ensure perfect stability as regards overturning, these

resultant thrusts must not only fall just within their respective points, as T_4 does, but sufficiently within them, leaving

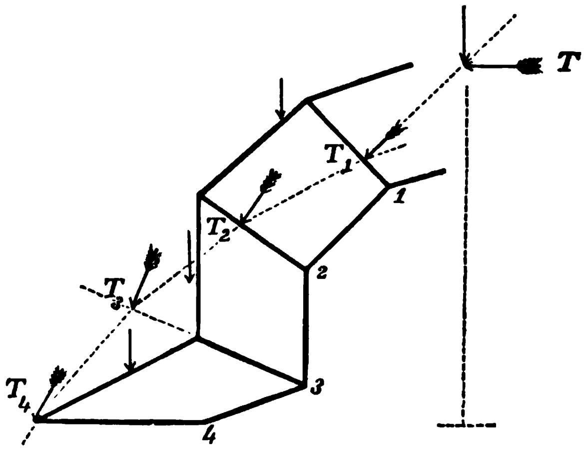


FIGURE 15.

a margin of joint. This margin or limit is variously given as $\frac{1}{8}$, $\frac{3}{10}$, $\frac{3}{8}$, or $\frac{1}{4}$, according to circumstances and partly according to choice, and is termed the modulus of stability. It will hereafter be deduced in value by analytical deduction, under the head of 'Limitation of Strain in an Arch-ring.'

2nd. *Sliding.*—At each joint in the series of blocks stability as regards sliding can be determined by resolving the thrust T before mentioned into two components—one N, normal to the joint, the other P, parallel to the joint. Then if F be the frictional resistance of which the joint is capable, putting $F = \mu N$, where μ varies between zero and infinity, to ensure equilibrium, F or μN must be greater than P; or $\mu > \frac{P}{N}$; that is, μ, the coefficient of friction, must be greater than tan δ; δ being the inclination of the thrust to the normal.

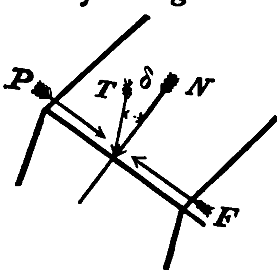


FIGURE 16.

3rd. *Crushing.*—The maximum stress on the material of which any block is composed, or on the substructure, must evidently be within the safe strain that the blocks can bear without crushing, and the foundations, &c., without subsidence. This will be applied in a solution for a bridge-pier hereafter.

Modes of Rupture in the Arch.—An arch may be considered as a loaded structure of cemented blocks of equal depth, symmetrical in form about a vertical axis passing through its crown, and fulfilling the conditions of stability already explained. In most cases also the loading is symmetrical with regard to the same axis. Its tendency to fracture will, however, vary with its form.

Experiment has shown that the tendency of the arch of circular curvature and of small rise is to break inwards at the crown and outwards at the haunches; this tendency may be counteracted by light loading at the crown and heavier loading at the haunches, or under similar loading by increasing the rise. With extremely flat arches the points of rupture are limited on the intrados to within its intersection with the pressure-curve, and on the extrados to below its intersection with the pressure-curve, at the haunches. These points of intersection are thus limits, or limiting points of rupture, and are easily determined by diagram when the pressure-curve has been obtained.

The pointed arch, having a very high proportionate rise, is liable to break inwards at the haunches and outwards at the crown under an excessive equally distributed load. This tendency may be counteracted by piercing or lightening the haunches and by heavily loading the crown. Also by reducing or altering the rise, some limit may be reached, when rupture will necessarily occur on the intrados at the haunches, or at the extrados at the crown, beyond

the points where the pressure-curve cuts these respectively. Such points of intersection are, then, the limiting points of rupture of the pointed arch.

Between the two extremes of a very flat segmental and a high-pointed arch there are many varieties of arch in form and in curvature, but under all intermediate conditions the same principles apply to determine the limiting points of rupture and the economic conditions of stability.

Hence, knowing the weak points of any arch through these means, it is merely necessary to locate the pressure-curve for a short distance near them; also when the arch is symmetrical in loading as well as in every other respect about a vertical axis through its crown, each half-arch will alone require these pieces of pressure-curve. But even if the two half-arches are not symmetrical, they may be separately treated.

The Location of the Pressure-Curve at the crown is a matter on which the location of every other part of that curve is dependent, and unfortunately is not practically determinable with much precision. The magnitude or amount of thrust at the crown is comparatively easily found; on the assumption that the thrust occurs within the depth of arch-ring, a condition of equilibrium that must obviously hold with rigid or comparatively non-elastic material, not extending beyond the ring at the crown.

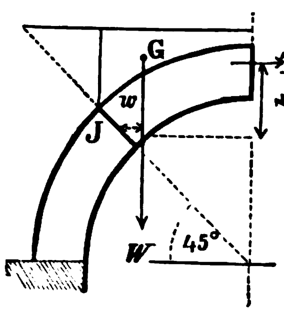


FIGURE 17.

The weight of arch affecting this thrust is the weight of it and its load down to a joint J inclined at 45° to the horizon; representing this by *W* (in the figure) collected at

G its centre of gravity which must be numerically determined in position, we have by equating moments about the point J,

$$T = \frac{W.w}{t}$$

where w and t are the leverages of W and T. But this assumes a knowledge of the value of t, whereas we actually know it merely within limits, which do not vary beyond the depth of arch-ring, under equilibrium.

Granting the amount and the location of the thrust at the crown; the following is the usual mode of obtaining the thrust at any other joint, which may be carried out either graphically or analytically.

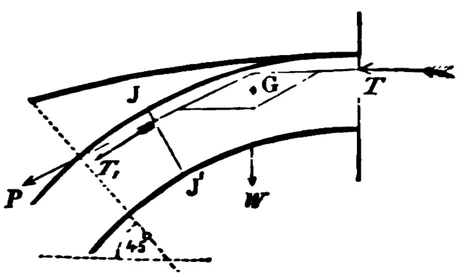


FIGURE 18.

Let T be the given thrust at the crown, JJ′ the joint at which the thrust is required; let W be the weight of the arch and its load from the crown to JJ′ acting at G the centre of gravity of the two. Then by simple resolution obtain P acting at G, and transfer it in its own direction to act at JJ′; now resolve P; its component T_1 normal to the joint is the required thrust. Conversely also if T_1 and W be known, T can be found; but it is more usual to obtain T through its moment equated with that of the weight of the whole arch and load down to some known joint of probable rupture. This with circular curvature is generally taken at 45°.

We can hence obtain two pressure-curves under the two extreme possible conditions, between which the actual pressure-curve must fall; and this is the utmost that the actual conditions allow.

The common practice of assuming that the crown-thrust acts exactly at the middle of the arch-ring is usually quite unwarrantable; but in *evident* cases of superabundant stability it can be assumed to act at the least favourable edge of the safer two-thirds of the arch-ring.

From the condition that the weight of the half-arch down to a joint inclined to the horizon at 45° alone affects the representative thrust of the half-arch, we can infer that the rest of the half-arch below that terminal joint may be treated simply as abutment; a principle that will be hereafter utilised. The same condition, applied to ordinary arches of circular curvature in connection with the experimental deductions about points of rupture, also involves the conclusion that these latter are generally located within 12° of the terminal joint.

It must be noticed that every arch may not only have to fulfil the conditions of stability when fully loaded, but also when unloaded; this therefore requires the determination of a separate pressure-curve, which should fall within the safer two-thirds of the arch-ring, and involves different limiting points of rupture.

This combination of the usual modes of treating arch-stresses is coarse, even when well applied; a really good method is still wanted.

The Panels of Metallic Arches.—A cast-iron arch consisting of a series of panels bolted together is, in the first place, treated generally as a rigid inelastic arch, and its pressure-curves are determined under the two extreme possible conditions of loading; in the next place the stresses on each separate panel have to be considered.

Thus, in the attached figure the thrust T_1 and the reaction A are given by the pressure-curve and through ordinary static resolution, their position and magnitude de-

pending on the load; but the upper side of the panel will act as a short fixed girder under an equably distributed load. The lower side of the panel must evidently present an amount of resistance equal to that of the upper panel as regards the uniform load, and besides bear such additional thrust-moments as the conditions may apply. But the strength of any two opposite sides of the panel must necessarily be balanced, so as to meet dispersed strain; hence the extreme stresses in either one will apply to the opposite one, in all calculations of stability.

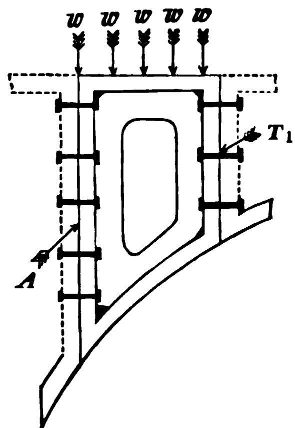


FIGURE 19.

Estimation of Stress on Abutments, Piers, Retaining-Walls, and Common Walls.—General Conditions.

A bridge-abutment, or a wall supporting a stone roof, are similarly situated with regard to stress when each of them

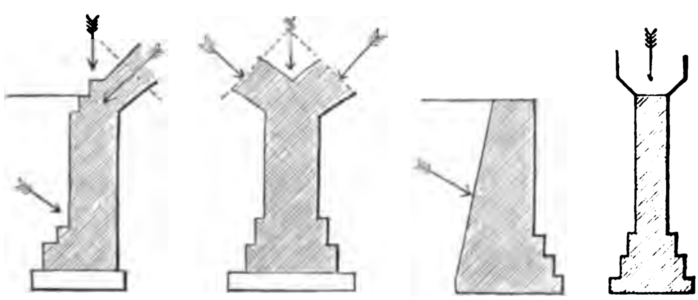


FIGURE 20.

is subject to thrust from an arch, as well as to stress from superincumbent weight.

A bridge-abutment supporting a girder or a truss corre-

sponds to a wall supporting a simple roof-truss; there is not any thrust, the whole stress being due to direct load and inherent weight. But a bridge-abutment may in either case also act as a retaining-wall in stress from earth-pressure, and in that case must resist either set of stresses independently of the other, being liable to these stresses independently during construction.

A bridge-pier between two arches is liable to thrust from each of them, and as during construction one arch may alone be standing at a time, the pier must be able to resist the single thrust from one independently of the other. But if the bridge-pier have merely to support girders or trusses there is not any thrust, and the stress is simply load and inherent weight. The same condition holds with a common house-wall.

A retaining-wall resists thrust due to earth-pressure on one side of it; and correspondingly a dam resists thrusts due to liquid pressure on one side only, in addition to supporting inherent weight.

Stress on Abutments of Bridges.—The abutment of a bridge may have merely to support a girder or truss in position, or to withstand the thrust of an arch, or of a curved rib, or in addition to give fixity to the end of a girder or rib. Taking the case when the adjoining span is bridged by a fixed curved rib (as more complicated, owing to the introduction of a moment of fixture), it will be supposed that the effect of earth-pressure behind the abutment, in aiding to resist thrust, is entirely neglected, as during construction such a condition may exist.

The external forces on the abutment are to be treated as acting at successive horizontal courses from the springing downwards, for the reason that masonry or brickwork is more usually thus built. It would, however, be more correct

constructively to place the courses at right angles to the line of thrust.

The resolution of the forces can be most conveniently effected when these are known quantities, evaluated from the conditions of the curved rib; as A and B the vertical and horizontal components of the thrust at springing, and $\frac{1}{2}Ql$ the rotary moment of fixture. With a simple arch this last force is absent.

Taking the horizontal axis of reference at springing level, and the vertical axis of y through the extreme edge of the footing of the abutment:

First; to obtain the stress at springing level, where $y=0$.

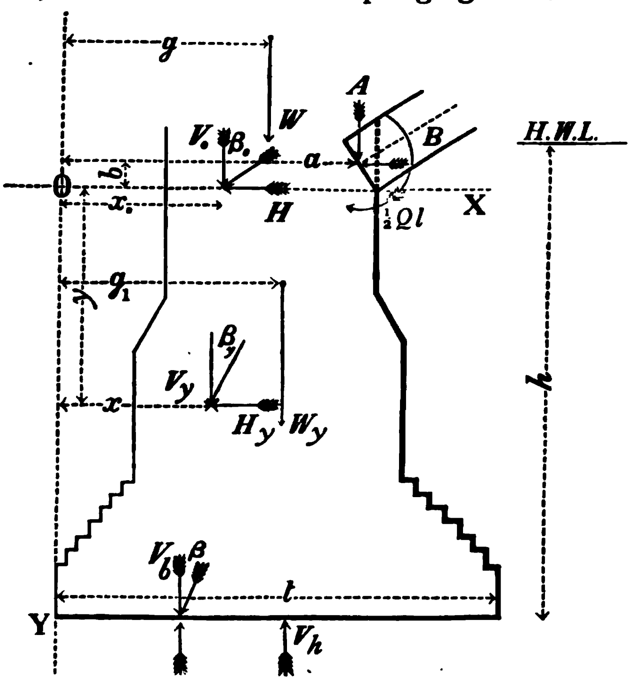


FIGURE 21.

Here putting W= weight of superstructure down to this axis;

g the distance of its centre of gravity from the axis of y;

β the angle made with the vertical by the resultant.

H = horizontal component of combined forces.

x_o the value of x to point of action of resultant.

V_o vertical component of combined forces.

we have

$$\left.\begin{aligned} V_o &= W + A\,; \\ H_o &= B\,;\ \text{and}\ \tan\beta = \frac{H_o}{V_o}\,; \\ V_o x_o &= Wg + Aa - Bb + \tfrac{1}{2}Ql. \end{aligned}\right\} \quad (1)$$

whence x_o can be obtained.

Next; to obtain the stress at any horizontal course, situated at a depth y below springing.

Here putting W_y = weight of abutment from springing down to y; and g_1 = the distance of its centre of gravity from the axis of y.

We have correspondingly

$$\left.\begin{aligned} V_y &= W + A + W_y \\ H_y &= B\,;\ H\ \text{being constant} \\ \tan\beta_y &= \frac{H_y}{V_y}\,; \\ x\,.\,V_y &= V_o x_o - B_y + W_y g_1. \end{aligned}\right\} \quad (2)$$

whence x may be obtained at any course between springing and foundation.

The same set of conditions will also yield results at the foundation course, V_b x_b &c., but these will be partial as a fresh set of external forces may here come into action; first the reaction from the soil, secondly water-pressure, due to possible infiltration under the foundation. There may hence be a second set of resultant thrusts and vertical forces acting upward; although no additional horizontal force is introduced.

Then if V_h be the vertical force due to the water pressure = weight of water having a volume equal to that displaced by the abutment up to extreme flood level $= tbh \times 62\cdot32$ in pounds; $= tbh$ in foot-weight;
where t is the thickness, b is the breadth of the abutment,
h is the height of flood-level above foundation course;
V' being the effective value of vertical force;

$$\left.\begin{aligned} V' &= V_b - V'_h \\ V'x' &= V_b x_b - V_h \cdot \tfrac{1}{2}t \\ \tan\beta' &= \frac{H}{V'} \end{aligned}\right\} (3)$$

whence x' may be obtained.

The consideration of strain throughout the whole abutment is necessarily dependent entirely on the stresses mentioned above that induce them.

The abutment having to resist crushing stress under extreme possible conditions; these are evidently, 1st. When A is greatest, or when the neighbouring arch or curved rib is fully loaded, and with the greatest moving load that can occur; also in the case of a curved rib when the temperature stress is highest. 2nd. When the moment of fixture $\frac{1}{2}Ql$ is the least or has the greatest negative value, which may occur when the nearest part of the neighbouring curved rib is unloaded, while the further part of it is loaded.

The stresses must therefore be applied under these extreme conditions.

Stress on Piers of Bridges.—The stresses are necessarily analogous generally to those on bridge-abutments before treated, but there are two sets of external forces, one from each arch or curved rib on each side, that may be unequal. Supposing for convenience that the springing of

each is at the same level, and taking rectangular axes as in the figure, we should have at springing—

$$\left.\begin{aligned} V_o &= W + A_1 + A\,; \\ H_o &= B_1 - B\,; \\ \tan\beta &= \frac{H_o}{V_o}\,; \\ V_o x_o &= Wg + A_1 a_1 + A\,a - B_1 b_1 + B b + \tfrac{1}{2}Ql - \tfrac{1}{2}Q_1 l. \end{aligned}\right\} \quad (1)$$

And at a depth y below springing, if W_y = weight of

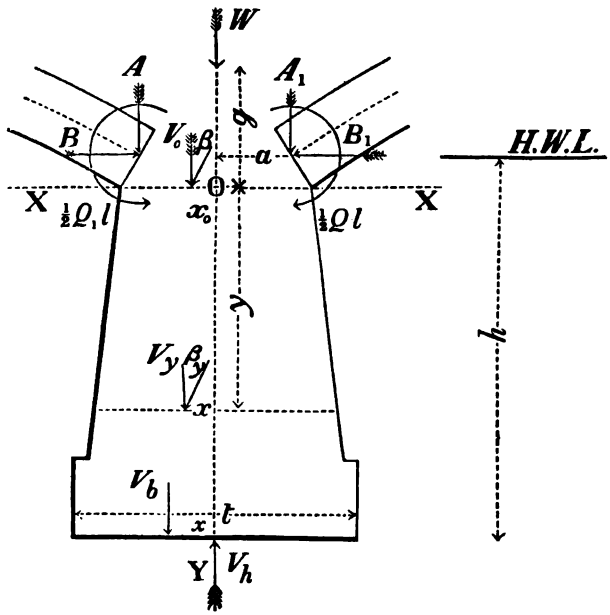


FIGURE 22.

pier from springing down to y, which is by supposition symmetrical in form—

$$\left.\begin{aligned} V_y &= V_o + W_y\,; \\ H_y &= B_1 - B\,; \\ V_y x &= \{ V_o x_o - (B_1 - B) y \}\,; \\ \tan\beta_y &= \frac{H_y}{V_y}\,; \end{aligned}\right\} \quad (2)$$

On arriving at the foundation course, if V_h is the verti-

cal force due to water-pressure = weight of water having a volume equal to that displaced by the pier up to extreme flood level,

V_b the value of V_y calculated for foundation-level,

V' the effective value of vertical force;

then
$$\left.\begin{aligned} V' &= V_b - V_h \\ V'x' &= V'_b x_b - V_h \end{aligned}\right\} \quad (3)$$

With unequal lateral forces B_1, B, it is sometimes more convenient to refer to a vertical axis at one edge of the pier, and to treat the case as the abutment in the last case. With balanced forces on the two adjacent arches or ribs, $x = 0$ throughout the pier.

The greatest stresses will occur when A_1 and A are greatest, or when both the adjacent ribs or arches are loaded to the extreme possible.

An unfavourable condition may occur during construction when one adjacent rib or arch is built, and the other is not placed; then only one set of forces, A, B, and $\frac{1}{2}Ql$, have to be dealt with as with an abutment. In some cases also a high temperature stress is combined with these.

An unfavourable condition will also occur after completion when $\frac{1}{2}Ql - \frac{1}{2}Q_1 l$ is greatest, which happens when a partial load exists on the neighbouring part of one adjacent rib and none on the neighbouring part of the other adjacent rib.

The stresses must therefore be applied under these extreme conditions.

Stress on Retaining Walls and Dams.

As the load on retaining walls and dams mostly consists in earth-pressure and fluid-pressure; expressions will be deduced to estimate the effect in resolved stress under

various circumstances and conditions, also to represent the effect of earth-pressure in equivalent fluid-pressure in some cases where this method can be conveniently adopted.

The plane of rupture for any mass of earth pressing against a wall is dependent on the natural slope that the earth would or might take, if unsupported and left to natural action for sufficient time. Earth-pressure is hence limited by the effect of the weight of the mass of earth included between the wall and the angle of repose or natural slope; and is thus dependent on the specific gravity of the earth as well as on its angle of repose.

(See Tables of Specific Gravity and of Natural Slopes.)

But the effect of the weight of this included mass is divided, one portion of it pressing against the wall, the other portion resting on the natural slope. The line of division between these two portions might be anywhere; but as the pressure against the wall is more important practically, its extreme position must be taken into account, and the line of neutral action or plane of rupture must be determined in accordance with this extreme.

Under these circumstances the cohesion of the earth may be neglected, and the friction exerted by it against the wall may be temporarily ignored; thus securing a safe result by over-estimating the stress.

The quantities determined in the following paragraphs, about earth-pressure are

1. The inclination of the plane of rupture, or of neutral action.

2. The amount of horizontal thrust induced by the earth-pressure.

3. The theoretical liquid-pressure that is equivalent to the earth-pressure.

Number 1. *Earth-pressure against a vertical wall, the earth being level to the top of the wall.*

The inclination of the plane of rupture.—In the figure let μ be the natural slope of the earth to the horizon,

α be the required angle as angle ABC made with the horizon by the required plane of rupture;

W be the weight of the mass of earth whose section is ACB and breadth unity;

w = the weight of a cubic foot of the earth;

h = the height of the wall;

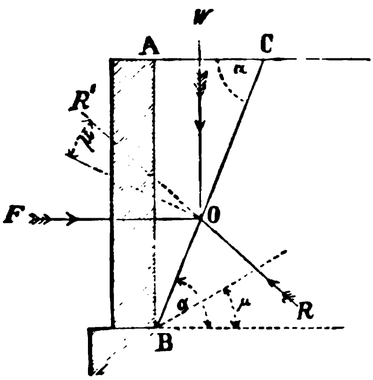


FIGURE 23.

R the resistance exerted by the bank at BC;

F the horizontal force, exactly sufficient to preserve equilibrium, exerted by the wall.

Then $W = \frac{1}{2} wh^2 \tan \alpha$.

Now the value of α will necessarily correspond to the maximum value of F, the horizontal force necessary to support the pressure exerted in turning the wall over, which is evidently more than enough to prevent its sliding. It is hence needful to determine the conditions under which F is a maximum.

By the parallelogram of forces acting at O

$$F = W. \frac{\sin \text{ROW}}{\sin \text{ROF}}. \text{ See Figure.}$$

But the resultant R is under such a condition necessarily inclined at an angle μ with the normal to the plane of

action, according to the principle of the inclined plane, and limiting angle of resistance; and by summation of angles, ROF $=90°-(\alpha-\mu)$, and ROW $=\alpha-\mu$; hence

$$F=W \,.\, \frac{\sin(\alpha-\mu)}{\cos \alpha-\mu}=W\tan(\alpha-\mu)=\tfrac{1}{2}wh^2.\text{cotg}\,\alpha\tan(\alpha-\mu);$$

differentiating this with regard to α, we have

$$\delta_\alpha F=\tfrac{1}{2}wh^2\{\text{cotg}\,\alpha\sec^2\overline{\alpha-\mu}-\text{cosec}^2\alpha.\tan\overline{\alpha-\mu}\}$$

and by reduction we obtain

$$\delta_\alpha F=\frac{w}{4}h^2.\frac{\sin 2\alpha-\sin 2(\alpha-\mu)}{\sin^2\alpha.\cos^2(\alpha-\mu)};$$

$$=\frac{w}{4}h^2.\frac{n}{d}; \text{ for convenience:}$$

differentiating this the second time, we get

$$\delta_\alpha^2 F=\tfrac{1}{4}wh^2.\frac{1}{d^2}\{d\delta_\alpha n-n\delta_\alpha d\}$$

But the conditions of the maximum are that $\delta_\alpha F=0$, and that $\delta_\alpha^2 F=0$; and since when $\delta_\alpha F=0$, $n=0$, we have

$$\delta_\alpha^2 F=\tfrac{1}{4}wh^2.\frac{1}{d}.\delta_\alpha n.$$

But the condition $\delta_\alpha F=0$ is satisfied by making α of such a value that $2(\alpha-\mu)=\pi-2\alpha$; applying this also to

$$\delta_\alpha^2 F=\tfrac{1}{2}wh^2.\frac{\cos 2\alpha-\cos 2(\alpha-\mu)}{\sin^2\alpha.\cos^2(\alpha-\mu)};$$

this expression becomes negative, or <0;

hence $\alpha=45°+\frac{1}{2}\mu$. . . . . Eq. I.

The angle formed by the natural slope with the vertical is hence bisected to obtain the required angle; and the sectional area of effective pressure is bounded by the wall and the plane of rupture thus determined.

The horizontal thrust.—The value of the horizontal thrust of the earth is equal in amount to F, but is exerted in an opposite direction;

$$\text{and as } F = W \tan(45^\circ - \tfrac{1}{2}\mu);$$
$$\text{and } W = \tfrac{1}{2} wh^2 \operatorname{cotg}(45^\circ + \tfrac{1}{2}\mu)$$
$$\text{hence } F = \tfrac{1}{2} wh^2 . \tan^2(45^\circ - \tfrac{1}{2}\mu). \quad . \quad . \text{ Eq. II.}$$

This Equation II. gives a value of F which represents earth-pressure (per unit of width) on any vertical plane extending from the surface of the earth down to a depth h.

Corresponding liquid-pressure.—In order to transform the earth-pressure on such a wall into liquid-pressure, it is merely necessary to imagine a theoretical fluid that has such a weight w_2 per cubic foot, so that

$$w_2 = w . \tan^2(45^\circ - \tfrac{1}{2}\mu)$$
$$\text{then } F = \tfrac{1}{2} w_2 h^2 \quad . \quad . \quad . \quad . \quad . \quad . \text{ Eq. III.}$$

and this is the expression for the fluid-pressure on the same surface per unit of width.

The following are values of $\tan^2\left(45 - \frac{\mu}{2}\right)$:—

μ°, . .	0	10°	15°	20°	25°	30°	35°	40°	45°	60°
Values,	1	·704	·589	·490	·406	·333	·271	·217	·172	·072

Notice that this method applies to any retaining wall with a vertical back, whatever its section may be.

Number 2. Earth-pressure against a vertical wall, with a surcharge of earth inclined at its natural slope.

The inclination of the plane of rupture.

Let μ be the natural slope of the earth, made with the horizon.

α the required angle for the plane of rupture made with the vertical.

W the weight of the prism of earth ABC whose width is unity.

w the weight of a cubic foot of the earth.

h_1 the total height from the foot of the wall to the level of the surcharge.

h_2 the height from the top of the wall to the level of the surcharge.

R the resistance exerted by the bank BC, or the resultant of W and F when equilibrium exists.

F the horizontal force, sufficient to preserve equilibrium, exerted by the wall.

Then $W = w\left(\frac{1}{2} h_1^2 . \tan a = \frac{1}{2} h_2^2 \operatorname{cotg} \mu\right)$.

And by the parallelogram of force under the conditions explained in the last paragraph Number 1,

$$F = W \operatorname{cotg}(a+\mu)$$
$$= \tfrac{1}{2} w . \left\{ h_1^2 \tan a - h_2^2 \operatorname{cotg} \mu \right\} . \operatorname{cotg}(a+\mu)$$
$$= \tfrac{1}{2} w . \frac{(h_1^2 \tan a - h_2^2 \operatorname{cotg} \mu)(1 - \tan a . \tan \mu)}{\tan a + \tan \mu} . \qquad \text{Eq. I.}$$

In order to find the conditions making F a maximum, and thence the value of a, let d represent the denominator of the last fraction, or $d = \tan a + \tan \mu$.

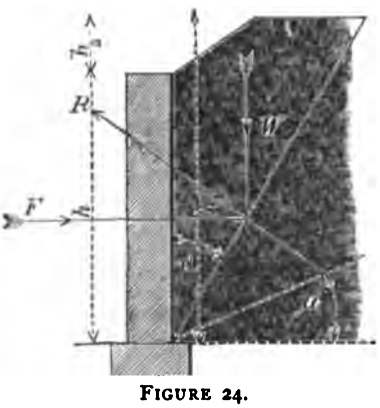


FIGURE 24.

Now the conditions giving a maximum value of F with regard to d will also hold with respect to a;

for $\quad \partial_a F = \partial_d F . \partial_a d,$

also $\partial_a^2 F = \partial_d^2 F . (\partial_a d)^2 + \partial_d F . \partial_a^2 d$

and as $\partial_a d = \sec^2 a$; $\partial_a F = \partial_d F \sec^2 a$.

But as $\sec^2 a$ has a finite value for all values of a less than 90°, $\partial_a F = 0$, when $\partial_d F = 0$.

Also, when $\partial_d F = 0$, $\partial_a^2 F = \partial_d^2 F (\partial_a d)^2$;

hence when $\partial_d^2 F$ is negative, $\partial_a^2 F$ is also negative.

Therefore the quantity d may be introduced into the expression for F last given, and the terms involving a may be replaced; as $\tan a = d - \tan\mu$; and by substitution and reduction we have

$$F = \tfrac{1}{2}w . \left[-dh_1^2 \tan\mu - \frac{1}{d}\sec^2\mu (h_1^2 \tan\mu + h_2^2 \operatorname{cotg}\mu) + h_1^2(\sec^2\mu + \tan^2\mu) + h_2^2 \right]$$

Differentiating this expression with regard to d, the first and second differential coefficients are

$$\partial_d F = \tfrac{1}{2}w \left\{ -h_1^2 \tan\mu + \frac{1}{d^2}\sec^2\mu (h_1^2 \tan\mu + h_2^2 \operatorname{cotg}\mu) \right\};$$

$$\partial_d^2 F = -\frac{w}{d^3} . \sec^2\mu \left(h_1^2 \tan\mu + h_2^2 \operatorname{cotg}\mu \right).$$

The first condition of the maximum gives a quadratic equation yielding

$$d = \pm \sec^2\mu + \frac{h_2^2}{h_1^2} \operatorname{cosec}^2\mu \Big]^{\frac{1}{2}}$$

The second condition is satisfied by any positive value of d. Using therefore the positive value of d resulting from the first condition, and replacing d by its original value $\tan a + \tan\mu$, we get

$$\tan a = \left[\sec^2\mu + \frac{h_2^2}{h_1^2} \operatorname{cosec}^2\mu \right]^{\frac{1}{2}} - \tan\mu. \quad . \quad . \quad \text{Eq. II.}$$

thus determining the limiting angle (a) of the plane of rupture.

The horizontal thrust.—Reverting to Equation I.

$$F = \tfrac{1}{2}w . \frac{(h_1^2 \tan a - h_2^2 \operatorname{cotg}\mu)(1 - \tan a \tan\mu)}{\tan a + \tan\mu};$$

it is evident that it is possible to obtain the expression for thrust in known terms, merely by substituting in it the

value of $\tan\alpha$, given in Eq. II.; some simplification is, however, advisable.

Noticing the expression for $\delta_a F = 0$ before given, we obtain—

$$h_1^2 \tan\alpha = \frac{\sec^2\mu\ (h_1^2 \tan\mu + h_2^2 \operatorname{cotg}\mu)}{(\tan\alpha + \tan\mu)^2}.$$

But in reducing the value of F after substituting for $\tan\alpha$ its value given in Eq. II., we may notice that the two first terms of the second member become equivalent to

$$-\tfrac{1}{2}w\left\{(\tan\alpha+\tan\mu).h_1^2\tan\mu + (h_1^2\tan\mu + h_2^2\operatorname{cotg}\mu)\frac{\sec^2\mu}{(\tan\alpha+\tan\mu)^2}\right. + \&c.$$

$$= -\tfrac{1}{2}w\left\{(\tan\alpha+\tan\mu).\,2h_1^2\tan\mu\right. + \&c.$$

$$= -\tfrac{1}{2}w\left\{2h_1^2\tan\mu\left(\sec^2\mu + \frac{h_2^2}{h_1^2}\operatorname{cosec}^2\mu\right)^{\frac{1}{2}}\right. + \&c.$$

$$= -\tfrac{1}{2}w\left\{2h_1^2\sec\mu\,(h_1^2\tan^2\mu + h_2^2)^{\frac{1}{2}}\right. + \&c.$$

This last value may therefore be substituted for the corresponding two terms before mentioned, and the value of F then becomes

$$\tfrac{1}{2}w\left\{-2h_1\sec\mu\,(h_1^2\tan^2\mu+h_2^2)^{\frac{1}{2}} + h_1^2\tan^2\mu + h_2^2 + h_1^2\sec^2\mu\right\}$$

or

$$F = \tfrac{1}{2}w\left\{h_1\sec\mu - (h_1^2\tan^2\mu + h_2^2)^{\frac{1}{2}}\right\}^2. \quad . \quad . \quad . \quad \text{Eq. III.}$$

a simplified expression for the horizontal thrust.

Corresponding liquid-pressure.—To transform this into corresponding liquid-pressure, let w_2 be the weight of a cubic foot of the theoretical liquid,

$$h = \text{the height of the wall} = h_1 - h_2$$

put

$$w_2 h = w.\left\{h_1\sec\mu - (h_1^2\tan^2\mu + h_2^2)^{\frac{1}{2}}\right\},$$

then

$$F = \tfrac{1}{2}w_2 h^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{Eq. IV.}$$

and this is the equivalent liquid-pressure on the same surface of vertical wall per unit of width.

Number 3. *Earth-pressure against a vertical wall with a surcharge inclined at any angle.*

In this case the method adopted in the last Number 2 is applicable, but the given inclination of the surcharge must be first substituted for μ in the expression giving the value of W; the values of a and of F can then be worked out in a similar way. Such a case is not of frequent occurrence, as the economy of wall-section effected by allowing any surcharge would be furthered by adopting an economic angle which would be that of the natural slope.

When the slope of the surcharge continues beyond a vertical axis drawn through the back of the wall, or the surcharge partly rests on the top of the wall, the superincumbent prism then forms no part of the section or prism of earth-pressure to be dealt with, and thus the value before used for W has to be slightly modified on this account. Also in considering the liquid pressure in such a case, it must be noted that h is not then $= h_1 - h_2$ as given in Number 2. In other respects the mode of calculation remains as given in Number 2.

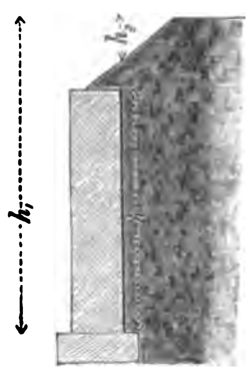


FIGURE 25.

Number 4. *Earth-pressure against an inclined wall, the earth being level to the top of the wall.*

The plane of rupture.—Let the back of the wall, AB, be inclined forwards, as shown in the figure, and let BC be the required position of the plane of rupture; BD a vertical.

Let μ be the angle of repose of the earth-surface or inclination to the horizon.

Let α be DBC, the required limiting angle of the plane of rupture made with the vertical.

Let β be ABD, the inclination to the vertical made by the back of the wall.

Let μ_2 be OFH, the angle made by the direction of F, with the normal to the back of the wall, or angle of repose for the surfaces of earth and of wall.

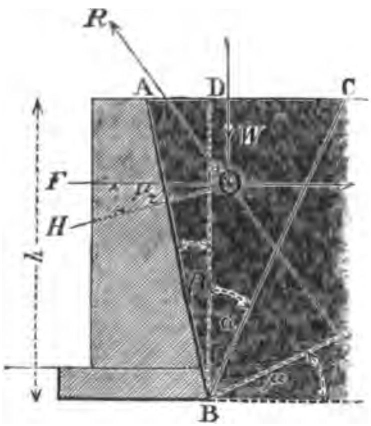


FIGURE 26.

W the weight of the prism of earth, ABC, whose width is unity.

w the weight a cubic foot of the earth.

h the vertical height of wall DB.

R the resistance afforded by the bank BC.

F the force exerted by the wall in the direction FO.

Then $W = \frac{1}{2} wh^2(\tan \alpha + \tan \beta)$.

Adopting a method corresponding to that of paragraph Number 2, we have in the parallelogram of forces,

$$F = \frac{W \sin \text{WOR}}{\sin \text{FOR}}.$$

But the resultant R is necessarily inclined at an angle μ with the normal to the plane of action; and the force F acts in a direction FO inclined at an angle μ_2 with the normal to the surface of the wall. And by summation of angles $\text{ROW} = 90° - (\alpha \mu)$; $\text{ROF} = \alpha + \mu + \beta + \mu_2$;

$$\therefore F = \frac{W.\sin(90^\circ - \alpha - \mu)}{\sin(\alpha+\mu+\beta+\mu_2)} = \frac{W.\cos(\alpha+\mu)}{\sin(\alpha+\mu+\beta+\mu_2)}$$
$$= \frac{\frac{1}{2}wh^2.\cos(\alpha+\mu)[\tan\alpha+\tan\beta]}{\sin(\alpha+\mu+\beta+\mu_2)},$$

or putting $\mu+\mu_2+\beta=\zeta$ for convenience,

$$F = \frac{1}{2}wh^2 . \frac{\cos(\alpha+\mu)[\tan\alpha+\tan\beta]}{\sin(\alpha+\zeta)} \quad . \quad . \quad . \quad . \quad . \text{ Eq. I.}$$

Differentiating this with respect to α, so as to obtain the conditions when F is a maximum; and putting $\delta_\alpha F=0$, and reducing, we obtain

$$-(\tan\alpha+\tan\beta).\cos(\zeta-\mu)+\cos(\alpha+\mu).\sin(\alpha+\zeta).\sec^2\alpha=0$$
$$-(\tan\alpha+\tan\beta)(1+\tan\zeta\tan\mu)+(\tan\alpha+\tan\zeta)(1-\tan\alpha\tan\mu)=0$$

hence

$$\tan^2\alpha+2\tan\alpha\tan\zeta-\tan\zeta\operatorname{cotg}\mu+\tan\beta(\operatorname{cotg}\mu+\tan\zeta)=0.$$

And the positive root of this quadratic (for a tangent) is

$$\tan\alpha=(\tan\zeta-\tan\beta)^{\frac{1}{2}}(\tan\zeta+\operatorname{cotg}\mu)^{\frac{1}{2}}-\tan\zeta \quad . \quad . \text{ Eq. II.}$$

If we differentiate again, finding the value of $\delta_\alpha^2 F$ and in it substitute this value of $\tan\alpha$, then $\delta_\alpha^2 F$ will be of negative value; hence this value of $\tan\alpha$ satisfies the condition of a maximum, and α will give the plane of rupture required.

The thrust on the wall.—The expression for thrust, F, may be obtained by substituting in Eq. I. the value of $\tan\alpha$ given in Eq. II.; some simplification is however desirable.

Observing that the term in Eq. I.

$$\frac{\cos(\alpha+\mu)}{\sin(\alpha+\zeta)} = \frac{(1-\tan\alpha.\tan\mu)\cos\mu}{(\tan\alpha+\tan\zeta)\cos\zeta}; \text{ in which}$$

$$1-\tan\alpha.\tan\mu = 1+\tan\zeta\tan\mu-\tan\mu(\tan\zeta-\tan\beta)^{\frac{1}{2}}(\tan\zeta+\operatorname{cotg}\mu)^{\frac{1}{2}}$$
$$=\tan\mu(\tan\zeta+\operatorname{cotg}\mu)^{\frac{1}{2}}\{(\tan\zeta+\operatorname{cotg}\mu)^{\frac{1}{2}}-(\tan\zeta-\tan\beta)^{\frac{1}{2}}\};$$

and

$$\tan \alpha + \tan \zeta = (\tan \zeta + \operatorname{cotg} \mu)^{\frac{1}{2}} (\tan \zeta - \tan \beta)^{\frac{1}{2}}.$$

Also that the term in Eq. I.,

$$\tan \alpha + \tan \beta = (\tan \zeta - \tan \beta)^{\frac{1}{2}} \{ (\tan \zeta + \operatorname{cotg} \mu)^{\frac{1}{2}} - (\tan \zeta - \tan \beta)^{\frac{1}{2}} \}.$$

Hence

$$F = \frac{wh^2}{2} \cdot \frac{\sin \mu}{\cos \zeta} \left\{ (\tan \zeta + \operatorname{cotg} \mu)^{\frac{1}{2}} - (\tan \zeta - \tan \beta)^{\frac{1}{2}} \right\}^2. \quad \text{Eq. III.}$$

the value of F in known terms, where $\zeta = \mu + \mu_2 + \beta$.

This expression may be reduced to the following form more suited to logarithmic computation:

$$F = \frac{wh^2}{2} \cdot \frac{\sin \mu}{\cos^2 \zeta} \left\{ \frac{\cos(\zeta - \mu)}{\sin \mu} - \frac{\sin(\zeta - \beta)}{\cos \beta} \right\}^2$$

according to the method of Moseley, but it must be noticed that this thrust is estimated as acting at a given angle with the normal to the back of the wall.

Corresponding liquid-pressure.—To transform this into corresponding liquid-pressure, let w_2 be the weight of a cubic foot of a theoretical liquid, so that

$$w_2 = w \cdot \frac{\sin \mu}{\cos^2 \zeta} \left\{ \frac{\cos(\zeta - \mu)}{\sin \mu} - \frac{\sin(\zeta - \beta)}{\cos \beta} \right\}^2;$$

Then $$F = \tfrac{1}{2} w_2 h^2, \quad . \quad . \quad . \quad . \quad . \quad \text{Eq. IV.}$$

h being the vertical height of the wall, as DB.

Number 5. *Earth-pressure on inclined walls generally.*

The effect of the inclination of the back of a wall and of the slope of its surcharge, when these occur together, is to greatly complicate the resolution of stress and cause much labour and risk of error. Such cases may

be partly assisted by graphic reduction and representing equivalent areas in different forms that only slightly modify the conditions without involving gross error.

One method frequently adopted with inclined walls is to reduce the case to one of a vertical wall by graphically rotating the back of the wall about a point at one third of its height, in a diagram on a large scale specially drawn for this purpose. This may be done so as neither to affect the amount of sectional area of the earth-prism nor the position of its centre of gravity.

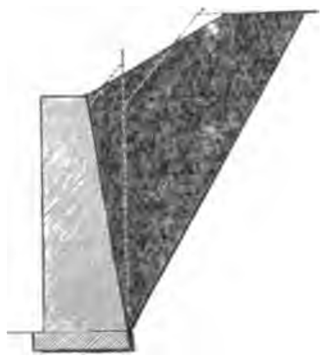


FIGURE 27.

Correspondingly also the surface-line of the earth-prism may be rotated or made to assume some new mean position, convenient for dealing with the surcharge, while compensation may be effected by slightly modifying or using some altered value for the height of the wall.

Very close approximations may be obtained by a judicious use of such methods, but some independent modification is generally advisable.

Number 6. *Liquid-pressure against walls and dams.*

The ordinary principles of the pressure of liquids are explained in all elementary works on 'Principles of Mechanics;' the following are common cases of application of those principles.

1. *Vertical dam or wall.*—Let A B in the figure be the vertical surface, shown in section, of a vertical dam opposing liquid-pressure; then its height A B = the depth of water = h.

Dealing with a section whose breadth is unity,

Let w=weight of 1 cubic foot of the liquid;

F=the liquid-pressure on the surface, concentrated at the centre of pressure.

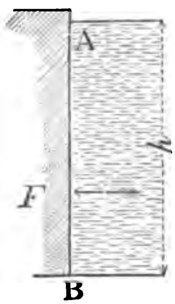


FIGURE 28.

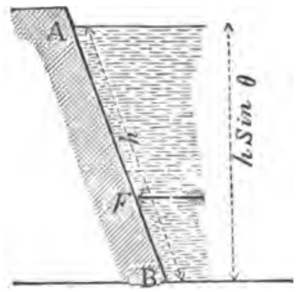


FIGURE 29.

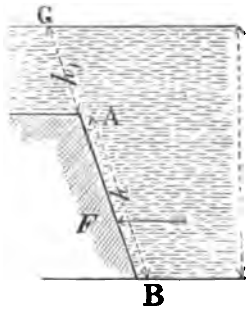


FIGURE 30.

Then $\frac{1}{2}h$ is the depth of the centre of gravity of the surface;

$$F=\tfrac{1}{2}\,wh^2;$$

and A F the depth of the centre of pressure is $\frac{2}{3}h$.

2. *Inclined dam or wall.*—Let A B in the figure be the sloping surface, shown in section, of an inclined dam opposing liquid-pressure; let A B=h, and let its inclination=θ, then the depth of water=$h \sin \theta$.

Dealing with a section whose breadth is unity,

Let w=weight of 1 cubic foot of the liquid;

F=the liquid pressure on the surface, concentrated at the centre of pressure.

Then the depth of the centre of gravity of the surface pressed=$\frac{1}{2}h \sin \theta$;

$$\text{and } F=\tfrac{1}{2}\,wh^2.\sin \theta;$$

and A F the depth of the centre of pressure is $\frac{2}{3}$ A B or is $\frac{2}{3}h$.

3. *Inclined dam when submerged, as a drowned weir.*— Using terms corresponding to those of the last case, see figure 30.

Let BA $=h$; AG at the same inclination h_1; θ the inclination; F the pressure on a surface whose breadth is unity.

The depth of the centre of gravity is $h+h_1 \sin\theta$;

$$F=\tfrac{1}{2}\, w\, (h^2-h_1{}^2) \sin\theta ;$$

and the depth of the centre of pressure is

$$\tfrac{2}{3} \cdot \frac{h^3-h_1{}^3}{h^2-h_1{}^2} .$$

In all such cases the prism of pressure has for its base an area equal to the surface pressed, and for its height the depth of the centre of gravity of the surface pressed. The centre of pressure is determined by elementary 'Principles of Mechanics,' in accordance with the form of surface.

Number 7. Liquid-pressure of semi-fluid masses; and representative liquid-pressure.

The ordinary conditions of actually existing semi-fluid masses that exert effect on structures are those of liquid mud and fine liquid sand; there is also a special condition of some soils under the influence of water, under which the greater portion of the mass remains nearly or partially solid, but is also imbedded in large quantities of liquid mud or water percolating around it; the whole of such a heterogeneous mass may then be considered semi-fluid.

This last-mentioned condition is that under which landslips are likely to occur. The effect of semi-fluidity has

to be guarded against by treating the pressure in two ways, first, as solid earth-pressure; secondly, as liquid-pressure with an enhanced specific gravity.

Representative liquid-pressure. — As in dealing with positively liquid mud, a liquid-pressure actually exists with a liquid of high specific gravity; so in dealing with ordinary dry soil or other materials we may for convenience assume an equivalent theoretic liquid-pressure to exist, that truly represents the pressure of the solid material. But in this case the specific gravity to be used is a theoretical quantity determined or reduced for the purpose from the terms and conditions of the case.

The theoretical weight per cubic foot used in such solutions—as in paragraphs Numbers 1 to 5—is hence merely representative, and is merely a means of reducing any pressures to their equivalent representatives in theoretic liquid pressure.

Pressure of Structures on Foundations.

Whatever means and appliances may be adopted for the construction of the foundation of a structure, and whatever the local circumstances and conditions of soil to be combated, the earth, whether rock or soil, has eventually to bear the structure with all its subterraneous or subaqueous additions that occur as foundations.

Assuming the ordinary practical conditions of a foundation, that the base is at right angles to the pressure from the structure, and the area of that base is made sufficiently large (either by spread of footings, inverted arches under openings, or other means), to bear the pressure with safety; the first point to be considered is the deviation allowed to the centre of resistance, or point traversed by the resultant pressure of the masonry or structure.

Admissible deviation.—The following is the formula for maximum deviation usually adopted in practice, which is to be found in Rankine's work, p. 378; it assumes that the pressure on the foundation is a uniformly varying compressive strain:

Let d=maximum permissible deviation of the centre of resistance from the centre of gravity of figure of the foundation;

S=the sectional area at the base of the foundation;

y=the distance of the centre of gravity from the edge of the base furthest from the centre of resistance;

I=the moment of inertia of figure of the base, computed as for the section of a beam relatively to a neutral axis traversing the centre of gravity at right angles to the directions of d and y;

b=the total breadth of the base in the same direction;

q=a coefficient dependent on I;

$$\text{then } d = \frac{I}{Sy} . = qb;$$

the above rule may be applied to foundations of every sort, whatever the soil may be, earth or rock.

Intensity of Pressure.—When the foundation of a structure is based on sound natural rock, or on rock covered with a layer of concrete, and thus formed to a homogeneous surface, the intensity of the pressure, or the pressure per unit of surface from the structure, should nowhere exceed one-sixth of the crushing pressure under which that earth or rock will yield, and should generally not exceed one-eighth.

When the foundation of a structure is to be based on earth which is of such a quality as would yield at the intended immediate base of the structure, deep or artificial foundations become necessary.

Deep foundations.—Granting that the base of the structure cannot be extended in area, there are three sorts of safe deep foundations obtained thus: 1st, by excavating below the originally intended base at all points until arrival at soil of the required condition; 2nd, by sinking a certain number of solid shafts to some stratum sufficiently firm to bear the load on the reduced area of support; 3rd, to drive piles or cylinders and trust to frictional resistance.

1. *Simple increased depth.*—In the first case, the opposing force that the soil is capable of exerting is evidently best demonstrated by the power that it shows naturally. A stratum of earth at a certain natural depth is capable of bearing the natural superincumbent earth; if then, as is usually the case, the base of the artificial foundation is arrived at by excavation and removal of soil, the weight of earth displaced in this process forms a guide to the weight of the structure that can be safely built on the base: the safe ratio that the one should bear to the other is given in the following equations due to Rankine, and this should not be exceeded.

Let w=weight of a cubic foot of the earth displaced.

h=depth of the foundation from the natural surface.

ϕ=the angle of repose of the soil.

p=mean intensity of pressure on the foundation or mean pressure per unit of surface.

p_1=greatest intensity of pressure.

p_2=least intensity of pressure.

b=breadth of base as before estimated.

d=maximum permissible deviation, before explained.

First, if the weight of the structure be uniformly distributed over the base.

Then $$\frac{wh}{p} > \left(\frac{1-\sin\phi}{1+\sin\phi}\right)^2$$

Next, if the weight of the building be so distributed that the pressure varies uniformly, the two following conditions are necessary :

$$\frac{wh}{p_1} > \left(\frac{1-\sin\phi}{1+\sin\phi}\right)^2; \text{ and } \frac{wh}{p_2} < 1.$$

From these also are deduced the following restrictions on the variation of intensity of pressure, and the maximum permissible value of *d*.

$$\frac{p^1}{p_2} < \left(\frac{1+\sin\phi}{1-\sin\phi}\right)^2; \qquad d = qb\,\frac{p-p_2}{p}.$$

Also when the figure of the foundation is symmetrical about its neutral axis, $p = \frac{1}{2}(p_1+p_2)$, and hence

$$\frac{wh}{p} > \frac{(1-\sin\phi)^2}{1+\sin^2\phi}; \qquad d = qb\,\frac{p_1-p_2}{p_1+p_2}.$$

The following are values of these functions of angles of repose :

ϕ	15°	20°	25°	30°	35°	40°	45°
$\left(\frac{1-\sin\phi}{1+\sin\phi}\right)^2$	0·346	0·224	0·165	0·111	0·073	0·047	0·0295
$\left(\frac{1+\sin\phi}{1-\sin\phi}\right)^2$	2·890	4·472	6·070	9·	13·62	21·15	33·94
$\frac{(1-\sin\phi)^2}{1+\sin^2\phi}$	0·514	0·421	0·283	0·200	0·137	0·090	0·057

These formulæ are taken from Rankine's book.

2. *Supporting shafts.*—When direct bearing is the object, shafts are sunk through a soft stratum until they arrive at rock or hard stratum ; such shafts when filled with concrete

or other material act simply as supporting columns, although aided, perhaps, to some extent by friction on their external surfaces. The resistance of the soil consists in direct opposition at all the horizontal areas or sections of shafts; the soil in itself exerting sufficient resistance to any further compression or crushing from the weight or pressure applied. The admissible deviation and intensity of pressure may be determined as in the former case of natural foundations.

3. *Frictional piles.*—When the foundation is purely frictional, the resistance to any insisting pressure or superincumbent weight is calculated according to the friction exerted by the external surface of the piles driven; neglecting the support afforded by their bases or sectional areas.

If the foundation be subaqueous, or in semi-liquid soil or mud, the power of flotation exerted by each pile, which is a force equal to the weight of water it displaces, may be taken into account, as well as the friction.

The friction itself of ordinary cylinders or piles in soil is empirically reckoned at from 8 to 16 foot-weight per square foot, or 500 to 1 000 lbs. per square foot of lateral surface of driven pile.

The condition least favourable to frictional resistance is that of very smooth iron cylinders in very soft ooze or mud, driven to small depths.

A pile may be said to be thoroughly driven when repeated blows amounting to a mechanical energy of 2 000 foot-talents, or 125 000 foot-pounds, produce an inappreciable effect in further driving, *i.e.* less than $\frac{1}{4}$ inch.

Rankine enters as follows into the relation between the blow required to drive an elm pile to a certain depth, and the greatest load that elm pile will then bear without sinking more; a relation between dynamic and static stress.

If W=the weight of the ram; h its height of fall;
y=depth to which the pile is driven by the last blow;
S=sectional area of the pile; l its length,
E=its modulus of elasticity,
P=the greatest load it will bear;

then as $$Wh=\frac{P^2 l}{4ES}+Py;$$

$$P=\frac{1}{l}\{(4ESW.hl+4E^2S^2y^2)^{\frac{1}{2}}-2ES.ly\};$$

but the units do not correspond throughout this formula. He says that P should be between 2 000 and 3 000 lbs. per square inch of S, the sectional area; and as the working load on elm piles is from 200 to 1 000 lbs. per square inch, this would give a factor of safety against sinking it between 10 in the first extreme to 3 in the last extreme.

The factor of safety for compression may be as high as $\frac{1}{8}$; but Rankine recommends $\frac{1}{10}$th for elm piles.

This principle may be applied to pile-driving generally, with alterations due to other conditions and materials.

NOTATION EMPLOYED IN STRESSES.

F	= external force or stress, representatively.
Ff	= moment of external force, representatively.
W	= a weight or load, representatively.
A, A_1, A_2, a_1, a_2	= vertical reaction at a pier or abutment.
B, B_1, B_2	= horizontal reaction at a pier or abutment.
V, v, &c.	= a vertical force or stress } at a section.
H, h, &c.	= a horizontal force or stress } at a section.
T, t, &c.	= a thrust on a section } at a section.
V_x, H_x, T_x	= values of V, H, or T, at a point whose abscissa is x.
V_{ex}, H_{ex}	= extreme possible values of V_x, H_x, &c.
w	= a weight-intensity, or weight either per unit of length, or per unit of surface.
l	= a length, a clear span of girder, or length of cantilever.
d	= a depth of any sort.
b	= a breadth of any sort.
c, c_1, c_2	= distances determining the application of a load or weight, or its limits.
O	= the position of an origin or of a middle point.
C	= the position of a centre of gravity.
x, y, z	= variables, or co-ordinates.
θ, ϕ, &c.	= variable angles, or unknown angles.
α, β, γ	= constant angles, or known angles.

Permanent Load.

Weight of Buckled Plate-flooring with and without angle iron.

For plates 3 feet square in all cases	Thick	Per sq. ft. gross	Per sq. ft. nett	Area to	Safe dead load per sq. ft.
	inches	lbs.	lbs.	sq. ft.	cwt.
Roofing and builders' plates .	0·048	2·30	2·25	1161	0·60
	0·066	3·14	2·62	855	0·95
	0·107	5·15	4·30	513	1·45
Flooring plates . .	$\frac{1}{8}$	6·00	5·00	441	2·22
	$\frac{3}{16}$	9·	7·50	297	5·55
Bridge floor plates . .	$\frac{1}{4}$	12·	10·	216	10·
	$\frac{5}{16}$	15·	12·5	180	14·9
	$\frac{3}{8}$	18·	15·	144	20·

Weight of Iron Roof-Trusses.

(From Unwin's 'Wrought Iron Bridges and Roofs.')

	Clear space in feet	Truss spacing in feet	Weight per square foot of covered area in lbs. Purlins	Principals	Total iron-work	Total with covering
Common trussed roofs - .	15	—	—	—	3·5	—
" " " .	37	5	1·1	3·5	4·6	6·9
" " " .	40	12	2·0	3·5	5·5	—
" " " .	54	14	6·5	3·0	9·5	—
" " " .	55	6·5	4·6	7·0	11·6	—
" " " .	72	20	4·2	2·8	7·0	—
" " " .	84	9	2·6	5·9	8·5	—
" " " .	50	10	"	"	3·0	5·2
" " " .	100	14	"	"	7·0	9·0
" " " .	130	26	0·8	5·6	6·4	8·0
" " " .	140	12	"	4·5	—	—
Bow-string roofs: Manchester . .	50	11	—	—	9·6	—
Lime Street . .	154	26	—	4·9	—	—
Birmingham . .	211	24	—	7·3	11·0	—
Curved-rib roofs: Corrugated Iron .	40	—	—	—	—	2·5
" " . .	60	—	—	—	—	3·5
Strasburg Railway .	97	13	—	—	12·0	—
Paris Exhibition .	153	26	9·5	5·5	15·0	—
Dublin . . .	41	16	3·4	7·3	10·7	—
Derby . . .	81·5	24	10·8	6·0	16·8	—
Sydenham . .	120·	24	7·9	3·9	11·8	—
" . .	72	24	8·4	2·9	11·8	—
St. Pancras . .	240	29·3	7·4	17·1	24·5	—
Cremorne Hall. .	45	14·5	6·2	5·3	11·5	—

Average Permanent Load on Road Bridges.

(From Unwin's 'Wrought Iron Bridges and Roofs.')

	lbs. p. sq. ft.	Total.
Timber road bridges—		
Planking and joists, single wooden platform	30	250
Stone or gravel roadway	100	
Traffic, a dense crowd	120	
Iron road bridges—		
Timber platform and ballast	90	230
Cross girders	20	
Traffic, dense crowd	120	
Iron road bridges with brick arches—		
Brick arches	48	340
Concrete and asphalte	42	
Metalling	118	
Cross beam	12	
Traffic, dense crowd	120	

For railway bridges, the dead load on cross girders, according to practice on various lines, is given in the following tables:

Average Dead Load in Tons per Foot of Track.

	Rails and fastenings	Sleepers	Ballast	Timber
English narrow gauge	0·03	—	·15 to ·21	0·10
Indian Lines 5½ ft. gauge	0·025	0·025	0·2	0·15
" " "	0·016	0·042	nil	0·03
Indian mètre gauge	0·013	0·014	0·2	—

Permanent Load. Weight of Roof-Coverings, &c.

	Flattest slope	Weight in lbs. per square foot of roof
Slating	$22\frac{1}{2}^{\circ}$	5 to 12
Slates and iron laths	$22\frac{1}{2}$	10
Pantiles	$22\frac{1}{2}$	10
Plain tiles	$22\frac{1}{2}$	20
Terraced roof 4″ on two 1½″ tiles, including joints 3″ × 3″, placed 3′ to 6′ apart	flat	100
Thatch	45	$6\frac{1}{2}$
Boarding ¾″	$22\frac{1}{2}$	$2\frac{1}{2}$
Boarding and sheet iron, 20 W.G.	4	$6\frac{1}{2}$
Wood framing for tiled and slated roofs	—	5 to 7
Iron " " "	—	3 to 10
Corrugated iron covering	4	3 to 4
Corrugated iron and laths	4	$5\frac{1}{2}$
Sheet iron 16 W.G. and laths	4	5
Sheet iron $\frac{1}{16}$-inch thick	4	3
Sheet zinc	4	1 to 2
Sheet lead	4	6 to 8
Sheet copper, ½-inch thick	4	1
King post-trusses, 20′ to 40′ span, framing, purlins, rafters, and collar-beam, the rise being one-fourth the span	—	7
Queen post-trusses, 40′ to 60′ span, framing, purlins, common rafters, and collar-beams, excluding weight of the beam in both cases	—	8 to 9

Average Rolling Load on Railway Bridges.

	Tons per foot of track.
On spans of 25 feet; English narrow gauge	2 tons
" 40 feet; " "	$1\frac{1}{2}$ "
" 60 and upwards; "	$1\frac{1}{4}$ "
On spans of under 20 feet; Indian 5½ ft. gauge	$2\frac{1}{2}$ "
" 20 to 30 feet; " "	2 "
" 30 to 40 feet; " "	$1\frac{3}{4}$ "
" 40 to 60 feet; " "	$1\frac{1}{2}$ "
" 60 and upwards; " "	$1\frac{1}{4}$ "
On spans under 4 mètres; Indian mètre gauge	2 "
" of 4 to 6 mètres; " "	$1\frac{1}{2}$ "
" of 6 to 15 mètres; " "	1 "
Add for each mètre in excess of 15	0·6 "

Traffic.—Load.

Men	Per foot of track	Weight of each	Per square foot
Unarmed men in crowd . .	—	160 lbs.	133 lbs.
Infantry in marching order .	—	200 lbs.	100 lbs.
„ in file, crowded . .	2 to 2½ cwt.	—	—
„ in fours, crowded . .	5 cwt.	—	—
Cavalry in file, crowded . .	1½ cwt.	12 cwt.	—
„ in half sections, crowded	3 cwt.	—	—

Animals	Maximum weight on one foot	Space below legs	Ground covered	Weight of each
Loaded elephants . .	44 cwt.	6½ ft.	100 sq. ft.	72 cwt.
Loaded camels . .	10 cwt.	4½ ft.	70 sq. ft.	15 cwt.
Loaded bullocks . .	3½ cwt.	3½ ft.	14 sq. ft.	6 cwt.
Cattle in drove, unloaded	—	—	9 sq. ft.	4 cwt.

Guns	Weight on fore wheel	Weight on hind wheels	Width of wheel track	Projection of carriage	Wheel base
	cwt.	cwt.	′ ″	′ ″	′ ″
7-inch B.L. rifled gun .	32	63	5 1	2 10	7 0
64-pounder „ .	31	78	5 3½	3 2	11 1
40 „ „ .	25	52	5 3½	3 2	11 3
20 „ „ .	15	30	5 2	0 7	9 3
12 „ „ .	15	20	5 2	0 7	9 0
9 „ „ .	14	16	5 2	0 8	8 9
13-inch siege mortar. .	22	76	5 3½	3 2	8 8
10 „ „ . .	19	37	4 4	1 10	8 2
1 „ „ . .	16	23	4 4	1 10	5 0

Locomotive Engines	Long	Wheel base	Weight	Weight on one driving wheel
	feet	feet	tons	tons
English average locomotive .	25	15	30	5
„ „ tank engine also	25	15	36	7½
„ „ tank engine .	30	15	45	—
French Great Northern . .	37	20	59	5
Indian lines, light engines .	24	14	25	3·5
„ „ .	41	15	24	4·0
„ heavy locomotives	27	15	34	6·3
„ „ „	55	20	32	4·0
„ mètre gauge locom.	21	10	16	—

Table of Weather Load for England.

	lbs. per square foot.
Rain absorption on roofs, maximum	5
Snow weight on roofs	5 to 20
Wind pressure, normal to surface as a maximum; average over a large surface	$31\frac{1}{4}$
Local maximum wind pressure, horizontally; limit adopted by the English Board of Trade	56

Table of components of wind-pressure for a given intensity of 40 *lbs. of horizontal pressure per square foot of vertical surface; blowing on surfaces of various slopes.*

Slope made with direction of wind	Intensities in lbs. per square foot		
	Normal	Components of normal pressure	
		Horizontal	Vertical
5°	5	4·9	0·4
10	9·7	9·6	1·7
20	18·1	17	6·2
30	26·4	22·8	13·2
40	33·3	25·5	21·4
50	38·1	24·5	29·2
60	39·2	21	34·0
70	40·9	14	38·5
80	40·4	7	39·8
90	40	0	40

The relation between wind-current, v, and resulting maximum wind-pressure, P, according to Hawksley; by formula $P = 0{\cdot}0765 . v^2 + 32{\cdot}2$.

v	P	v	P	v	P
feet per sec.	lbs. per sq. ft.	feet per sec.	lbs. per sq. ft.	feet per sec.	lbs. per sq. ft.
10	0·24	60	8·55	110	28·75
20	0·95	70	11·64	120	34·21
30	2·14	80	15·21	130	40·15
40	3·80	90	19·25	140	46·57
50	5·94	100	23·76	150	53·46

CHAPTER III.

RESISTANCE AND STRAIN.

TAKING the general representative equations of the stress and strain, already treated as fundamental principles in Chapter I., namely—

$$\Sigma F_1 = \Sigma R_1 ; \qquad \Sigma F_1 f_1 = \Sigma R_1 r_1 ;$$
$$\Sigma F_2 = \Sigma R_2 ; \qquad \Sigma F_2 f_2 = \Sigma R_2 r_2 ;$$
$$\&c. \qquad \&c.$$

we have already treated of the forces, loads, and stresses that occur on the first side of these equations, it now remains to enter into the resistances, and internal strains represented on the other side.

These resistances or strains are called into action by the loads and stresses resolved at the point or section under consideration, but are actually exerted by the properties of the material strained. The evidences of deficient resistance are molecular alteration, and distortion of general form; when rupture occurs, the ultimate resistance has been reached, and the set of conditions of stress and strain have arrived at a climax.

The *ultimate* resistance or strength of a material or of a body is measured by the stress producing either rupture or utter ruin of the body strained. The tabulated experimental values of ultimate resistances of various materials are the intensities of the breaking stresses that have been

brought to bear on them in various ways; and are termed Moduli of Strength.

It is generally assumed that in these experiments the direction of grain or fibre of material has been so disposed as to be most favourable to strength.

The *proof* resistance or proof-strength of a material or of a body is measured by the highest stress it is capable of withstanding without permanent injury, impaired strength, or evidently-damaging alteration of form or of molecular set. This is also generally expressed by saying that the proof-strain must not exceed the elastic limit; or that after removal of the proof-strain the body must perfectly recover its original form; yet the existence of a very slight set is not considered injurious with some materials.

The experimental determination of intensities of proof resistance affords numerical results that are in many cases liable to much uncertainty and inexactitude; the principal object in proof-testing is to afford evidence of sufficient margin of security for the piece tested, when subsequently employed to resist less strain.

The *safe* resistance, represented by the stress that the material can constantly bear with perfect safety, is fixed by ratio to its ultimate resistance; this ratio is termed the ratio or coefficient of safety, and varies from about $\frac{1}{2}$ to $\frac{1}{10}$ for various materials under different modes of applying the load or stress. The values of these coefficients of safety are purely empirical, and in application require some judgment; their numerical values are tabulated in the accompanying tables of resistances of material.

The *actual or working* resistance of a body or material is the strain induced by the stress actually applied to it, whatever it may be; in structures this strain should not

much exceed the empirical safe resistance, and should never exceed the proof-strength.

Resistances or strains are of the following sorts:— 1. Tensile; 2. Compressive; 3. Shearing; 4. Torsive; 5. Transverse; 6. Flexile; according to the condition and position of the material strained with regard to the insistent stress.

The term transverse strain is used to denote collectively the sum of the strains induced by a transverse stress or load causing tendency to rupture. Flexile strain which occurs in the flexible condition of a body is also a compound strain, induced by transverse stress, although the resisting power expressed by the single word stiffness is different from the former, being a mere resistance to deflection.

Conditions of stiffness are treated and determined separately from those of strength and of stability dependent on strength; whether the stresses are treated collectively, or resolved into components.

Elasticity is the power exerted by a body or a material in recovering its original form on the discontinuance of stress of any sort; hence it is of many kinds. Sensibly perfect elasticity is a vague term denoting that any set produced is very small and unimportant. Moduli of elasticity are experimental values of constant intensities of elastic power for the same material, under the condition of sensibly perfect elasticity; elasticity after tension and elasticity after compression are usually indicated by the same modulus, or intensity per unit of length; elasticity after torsion is indicated by a second modulus: these two moduli are alone met with in tabulated experiments of common use.

The equation of stiffness, through which the alteration of form of a body under stress is determined, is dependent

on moduli of elasticity; such determinations are practically limited to the comparatively small deflections that occur in bodies whose lengths are proportionately in considerable excess of their widths and depths.

Resilience is the amount of work performed by the elastic power of material in a body in recovering its original form on discontinuance of stress.

Resilience is estimated by the total work performed by the stress on a body from the moment of first application until any known strain is produced; this total amount of work being, on discontinuance of stress, expended in resilience, or spring.

The proof-resilience similarly corresponds to the proof-stress; and the corresponding values of moduli of resilience for various materials are equal to the squares of the respective proof-stresses divided by the respective moduli of elasticity.

There are as many kinds of resilience as there are combinations of sorts of elasticity and sorts of stress that can occur together; but among these, only three, resilience after tension, after compression, and after deflection, within elastic limits, have received much attention.

Returning to the subject of strains before enumerated.

Tension.—A stress inducing tensile strain, or putting material in a state of tension, has a tendency to stretch it; any set produced is an elongation. In many cases the weight of the material strained forms part of the total stress, acting in the direction of the axis of figure of the body.

The law of tension, holding good for nearly homogeneous material, is that the tensile resistance or strain is proportional to the cross-section of the body strained.

Hence the general equation of safety under tensile strain is thus:

Let R=safe tensile resistance of the material per unit of section, obtained by applying a suitable coefficient of safety to the modulus of tenacity.
S=sectional area strained.
F=the force, or stress applied.

Then the limiting value of F, that is consistent with safety is $F=S\,R$; . . . . (1)

Similar units of weight must evidently be used on both sides of this equation (as well as in others generally), and similar units of surface must be used in the value of A and in that of R; whatever they may be is immaterial.

The set, stretch, or elongation of the body produced by any stress less than the proof-strain is dependent on the tensile elasticity of the material, and is thus obtained:
Let E be the modulus of tensile elasticity of the material per unit of section,
l_1=the elongation, and L=original length of the body,
F=the external force, or stress applied,
S=the sectional area strained,

then $l=L\,.\,\frac{F}{S.E}$ . . . . (2)

When F is equal to the proof-strain, the value of L corresponding is the proof-set or proof-elongation.

Secondly, if the body be suspended vertically while in tension, so that its total weight aids in producing strain:
Let L=original length of the body,
L_1=stretched length of the body,
and $L_1=L$ is the required elongation,
w=weight of the body per unit of length,

F_1 = the stress applied independently of w, S and E being as in the last case;

$$\text{then } L_1 = L\left(1 + \frac{F_1}{SE}\right) + \frac{w}{2SE} \cdot L^2 \quad \ldots \ldots \ldots \quad (3)$$

Similar units of length, of weight and of surface must be used throughout this equation (as well as in others generally), their values are immaterial.

Resilience after tension, in a bar or rod:
If F = the stress applied gradually,
E = the modulus of elasticity per unit of sectional area,
L and S the length and sectional area of the body.
U = the resilience exerted;

$$\text{then } U = \frac{F^2.L.S}{2E}; \quad \ldots \ldots \ldots \quad (4)$$

Also if F_1 be a stress applied suddenly, the corresponding resilience is $U_1 = \dfrac{F_1^2 . LS}{E}$. . . . . (5)

Compression.—A stress inducing compressive strain in a material has a tendency to crush it; the set produced under perfect mathematical conditions of application and perfect uniformity of material is a shortening.

In many cases the weight of the body strained forms part of the total stress acting in the direction of axis of figure of the body.

The law of compression, under the above-mentioned conditions, is that the resistance or compressive strain is proportional to the cross-section of the body strained.

Hence the general equation of safety under compressive strain corresponds to that under tensile strain, thus,

If R = safe compressive resistance of the material per unit of section obtained by applying a suitable coefficient of safety to its modulus of compression,
S = sectional area strained,
F = the force or stress applied,

then the limiting value of F, that is consistent with safety, is $F = S R$; . . . . (6)

The set or shortening of the body produced by any stress less than the proof-strain is dependent on the comprehensive elasticity of the material, and is thus obtained under the same rigidly perfect conditions, for the three following cases.

First, if the body is comparatively weightless,

let E = the modulus of compressive elasticity of the material per unit of section,
L = the original length, and l = the set,
S = the sectional area strained,
F = the external force or stress applied,

then $l = L \cdot \dfrac{F}{S E}$ . . . . (7)

When F_1 = the proof-strain, then l_1 = proof-set.

Second, let the body be placed vertically, having a weight w acting in the same direction downwards as the external force F;

then if L_1 = the reduced length

$$L_1 = L\left(1 - \frac{F}{S E}\right) - \frac{w}{2 S E}; \quad . \quad . \quad . \quad . \quad (8)$$

Thirdly, if the weight acts downwards, but F acts upwards, F is relatively negative,

$$\text{and } L_1 = L\left(1 - \frac{F}{SE}\right) + \frac{w}{2SE}; \quad \ldots \ldots \quad (9)$$

in all cases the set $l = L - L_1$.

But if the above-mentioned conditions of perfect homogeneity of material and perfectly exact directions and bearing of stress do not hold, the set produced may be lateral, as shown in buckling, bulging, splitting, or shearing. That is to say, the compressive strain unequally induced or unequally distributed over the cross-section, partly resolves itself into flexile, lateral, or shearing strains before any set is caused.

Hence the comparatively simple case of a pillar or stanchion supporting simple weight may involve the consideration of complicated strains, or may force a recourse to empirical formulæ for strength and stability.

When the modulus of compressive elasticity for a material is not known, it is usual to apply its modulus of tensile elasticity to the same purpose. Some persons even believe they are identical generally in most materials.

Shearing strain.—When treating of stresses due to transverse load, it is usual to resolve them into two sets, the strictly lateral stresses and the lengthway stresses (on a horizontal beam the former are vertical, the latter horizontal; on a vertical column the converse). That method does not assert that there are not any oblique stresses due to the action of a shear-couple, for there are such, though little is known about them; but it renders them theoretically evanescent.

The lateral stresses hence act with varying values in parallel lines at infinitely small distances at every cross-

section between the points of action of the two forces constituting the shearing couple. The strains directly induced by these lateral stresses are shearing strains or resistances to sliding movement in the lateral planes.

In many cases, owing to the mode in which material is practically distributed, shearing resistances are cohesions across-grain of material, as in bolts, trenails, rivets, mooring posts, and in webs of girders, depths of beams, &c.

Resistance to shearing is estimated per unit of sectional area under lateral shear; the equation of safety therefore corresponds to that with compression or tension.

If R is the safe shearing resistance, obtained by applying a suitable coefficient of safety to the modulus of shearing resistance. Then $F=SR$,

or more frequently $Ff=SR$ . . . . (10);

where Ff is the moment of lateral stress on the section.

As there may be a series of such stresses or stress-moments along a continuous body as a beam or a column, there is a corresponding series of strains induced, and these can then be treated graphically. In framed or braced roofs and girders there is not a direct shearing resistance offered at every section; but it is merely afforded at detached points through the bracing, which supplies the want of continuous sectional resistance.

The set or distortion under shearing stress of a square prism, is thus represented:

If l=the measure of distortion, hereafter explained,

F=intensity of stress over the four faces,

E=modulus of shearing elasticity,

$$l=\frac{F}{E} \quad \ldots \ldots \ldots \ldots \ldots \quad (11)$$

And, if d=diagonal of the prism, originally,

d_1=elongation of diagonal,

d_2=contraction of diagonals,

$$l=\frac{d_1+d_2}{d}.$$

Torsion.—A torsive strain, induced by the stresses due to a torsion-couple, is the resistance to angular displacement, or twisting around an axis, at a section of material. It occurs in the cylindrical shafts of machinery and mechanism ; a familiar example occurs in attempting to work a ship's capstan, when the check or brake is applied ; but it is seldom markedly exemplified in structures.

As torsion is a modified shearing, the modulus of resistance to shearing of any material is commonly used as a modulus of resistance to torsion ; and a safe resistance is obtained by applying a coefficient of safety.

With prismatic solids, the equation of safety may be thus expressed in a few simple cases :

Ff=moment of torsive stress,

θ=angle of torsive displacement,

L=length of solid body strained,

R=safe resistance of the material to torsion,

I=moment of inertia of the section about the axis,

then $$Ff=\frac{R.I}{L}.\theta. \qquad (12)$$

With a cylinder of radius (r)—

$$Ff=\frac{R.\pi r^2}{2L}.\theta; \qquad (13)$$

with a rectangular prism, sides (a) and b—

$$Ff=\frac{Ra^3b^3}{3L.(a^2+b^2)}.\theta; \qquad (14)$$

with a shaft of varying diameter, d_1 and d_2 being the two extreme diameters under stress,

$$Ff = \frac{3R.\pi.\theta}{2L}.(d_1 - d_2)\,(d_2^3 - d_1^3) \quad . \quad . \quad . \quad (15)$$

In a structure a beam may be centrally loaded with reference to a cross-section, but the supporting force F may act extra-axially at a distance f from the axis; then the twisting stress, represented by the moment Ff, should be taken into consideration when determining the strength of the beam.

The usual equation connecting the twisting stress-moment with the torsive elasticity is

$$\Sigma\, Ff = \frac{2.E\,I.\pi}{L}\,; \quad . \quad . \quad . \quad . \quad . \quad (16)$$

When there is both an end thrust F and a torsive moment M on a length of shaft L,

$$\frac{F}{EI} + \frac{M^2}{4\,(EI)^2} = \frac{\pi^2}{L^2}\,;$$

Transverse Strain.—The effects induced by a load or stress applied laterally or transversely to a homogeneous column or beam of uniform symmetrical section are necessarily compound strains of various sorts, but may under some special conditions be treated as a single strain. The resistance to cross-breaking is then a collective simple resistance for purposes of calculation.

This principle, however, as in the case of flexile strain, can only apply to beams or columns of a similar kind, on which all the forces and reactions are similar; it is safer also to further confine it to cases when the load or stress can be representatively applied at the middle of the length; and it is safest at present to avoid it entirely.

Supposing the material to be comparatively weightless,

Let R be the safe resistance per unit of section of the material to cross-breaking within elastic limits, obtained by applying a suitable coefficient of safety to the modulus of resistance to cross-breaking; let I be the sectional moment of inertia, l, b, and d be the length, breadth, and depth, and F the force, stress, or weight applied at the middle of the free beam; then the equation of safe strain and stress is

$$F = \frac{8IR}{d.l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

(It must be noticed that some tabular values of coefficients of resistance to transverse stress or of cross-breaking, which are also termed moduli of resistance, are values of $\frac{1}{18} R$ here used. Also that many of the tabulated values of either sort given by various writers are not the results of direct experiment, but are computed by applying some coefficient to the tensile resistances of the same materials, such coefficients being due merely to the class of material, as stone generally, grouped classes of timber, &c., but not to the actual material.)

There is, however, some doubt whether the general formula would hold for sections other than the rectangle, square, ellipse, and circle.

With a rectangular section, which is the form that has been most investigated, if b be the breadth and d the depth, and the beam be free as before, the above becomes

$$F = \frac{2bd^2.R}{3l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

but if the beam be fixed at both ends

$$F = \frac{bd^2.R}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Correspondingly also with a cantilever of uniform

rectangular section, having the stress F applied at the free end.

$$F=\frac{1}{6}\frac{bd^2.R}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Flexile Strain.—When a homogeneous solid free beam or column of uniform symmetrical section is subjected to lateral stress, a flexile strain or resistance to deflection is induced, which varies with different materials, but is capable of being dealt with within elastic limits.

Flexile strain is necessarily compound, being composed of compressive, tensile, and shearing strains; but under limited conditions, as above described, it may be treated, independently of such resolution, in the same way as a simple strain.

Supposing the material to be comparatively weightless, and the lateral stress to be represented by a single stress F applied at the middle;

then if $l=$ the length of the body,

$I=$ its sectional moment of inertia,

$R=$ the flexile resistance per unit of section of its material within elastic limits;

the safe equation of stress and strain may be represented generally thus:

$$F=\frac{I.R}{l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

all dimensions being taken in similar units, and a similar weight unit being employed in F and in R.

There is, however, much doubt whether this general formula would hold for sections other than the rectangle, square, ellipse, and circle.

With a rectangular section, which is the form that has been most investigated, of depth d, breadth b,

$$F=\frac{bd^3}{12l^3}.R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and it also has been empirically determined that in this case if ξ = the flexure or maximum deflection produced, and E = the modulus of *tensile* elasticity of the material,

$$R=\frac{E}{3\xi};\ \text{or}\ \xi=\frac{E}{3R};\quad . \quad . \quad . \quad . \quad . \qquad (23)$$

Correspondingly, also, with a cantilever of uniform rectangular section, having a length l, and a stress F at its free end,

$$F=\frac{bd^2 R}{192\,l^2};\ \text{and}\ \xi=\frac{E}{3R};\quad . \quad . \quad . \quad . \quad . \qquad (24)$$

It is unfortunate that experiments have not been conducted with the view of obtaining either R flexile strain or flexile elasticity, but that experiment has aimed at getting mere deflection under very limited conditions with the help of a known modulus of tensile elasticity. (See works of Barlow, Hart, Sankey, Lang, &c., on properties of timbers.)

Remarks.—In all cases when the method of treating either flexile strain or transverse strain as simple collective strain is adopted, the representative stress or load is supposed to act at the middle with beams, and at the free end with cantilevers, and all other load or stress must be reduced to such a representative stress. It is further necessary that the materials to which this method is applied should be nearly isotropic, or that their tensile and compressive elasticities should not be widely different. Practically this method is applied to joists, short beams, small girders, &c., in house-building; but in engineering and in large structures the mode of resolving the strains into component simple strains is commonly adopted, which will be afterwards explained.

Units.—In all foregoing formulæ it is assumed that

similar and corresponding units of length, weight, pressure, and surface are used throughout an equation.

For example, if a safe resistance R for a material is obtained by applying a coefficient of safety as $\frac{1}{8}$th to U the ultimate resistance of the same material given in a table: and if it be predetermined to use pounds, inches, and square inches, then R must be in pounds per square inch; F, the stress, must be in pounds; dimensions b, d, l, &c., must be all in inches; S, any sectional area must be in square inches. If it be predetermined to use hundredweights and feet, the value of R must be reduced to hundredweights per square foot; F will be in hundredweights; dimensions and sections will be all in feet and square feet respectively.

The formulæ have been symmetrically arranged so as to suit any units, but those units must be adhered to. In exceptional cases where there is any deviation from this rule, as in borrowed equations, special mention is made.

Resolved Transverse Strains.

Transverse loads or stresses on beams, girders, columns, &c., may be resolved into lateral stresses, which are resisted by shearing strains, already treated, and into lengthway stresses, or longitudinal stresses, which are of a compound nature, and induce compound strains.

The total lengthway stresses have been already treated in Chapter II.; the strains induced by them are compressive on one side of some neutral axis in a horizontal beam or girder, and tensile on the other side. It hence is necessary to limit and collect these strains, or if possible to express them in representative strain of either sort independently.

An equation of stress and strain will evidently hold at every section along the beam or continuous body.

In every section to be dealt with, there will necessarily be a neutral axis where the strain will be zero, neither compressive nor tensile, and a neutral plane will thus exist throughout the whole length and the whole width of the beam under transverse stress, while the amount of strain at any sectional lamina will vary with the distance of the lamina from the neutral axis. Hence the strains and corresponding stresses are uniformly varying in the respective strain-prisms of tension and of compression.

Now although the position of the neutral plane is yet unknown, we may temporarily assume it to be somewhere within the middle third of the depth of the beam, and take an axis on it, passing through the width of the beam, as an axis of reference for strain moments.

Integrating the compressive strains, their moment

$$M_1 = \frac{R_1 I_1}{a_1};$$

and similarly integrating the tensile strains, their moment

$$M_2 = \frac{R_2 I_2}{a_2};$$

where R_1 and R_2 are the safe resistances to compression and to tension respectively of the material,

I_1 and I_2 are the respective moments of inertia of the two separate portions of sectional area compressed and stretched;

a_1 and a_2 are the respective distances from the neutral axis of the extreme edges of those two areas, termed extreme axial distances, or more simply, axial distances;

M_1 and M_2 are the moments of sectional resistance of the two areas. (See Table of Moments of Sectional Resistance at the end of this chapter.)

But the whole resistance or strain of the whole section is the sum of the total compressive and total tensile strains, because the rotation movement has a similar tendency or is in the same direction. Therefore the total safe resistance afforded by the section is

$$M_1 + M_2 = \frac{R_1 I_1}{a_1} + \frac{R_2 I_2}{a_2};$$

and if H be the horizontal stress brought to bear on this section; see Stresses, Chapter II., p. 18,

$$H = M_1 + M_2 = \frac{R_1 I_1}{a_1} + \frac{R_2 I_2}{a_2}; \quad . \quad . \quad . \quad (25)$$

which is the equation of safety holding between the horizontal stress and the strains induced at any section.

This is the usual mode of estimation through the resistance of strain prisms, but it evidently under-estimates the total resistance in the prism, by neglecting the adherence of the laminæ to each other. In the attached figures the position of a set of resisting laminæ is shown; if their adherence be neglected, each lamina is an independent bar, and the total resistance is sectionally represented by a triangular prism; but if adherence be allowed for, the inner laminæ take up some of the strain on the outer laminæ, and the total resistance is a prism of curved section. The curve must be found for internal coherence of each material; but experimental data for determining it are still wanting. In the interim the old method, though faulty on the safe side, must be retained.

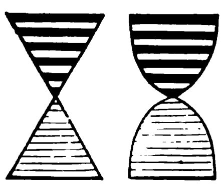


FIGURE 31.

Yet it now remains to determine the true position of the neutral plane in various cases; otherwise I_1, I_2, a_1, a_2, hitherto representative terms, could not be computed.

First let the material be nearly isotropic, having practically equal resistance to tension and compression; then $R_1 = R_2$, also with the corresponding elasticities $E_1 = E_2$, and we may deduce that

$$\frac{R_1}{a_1} = \frac{R_2}{a_2}.$$

Also we find that the neutral plane passes through the centre of gravity of the whole section.

In this case, too, as $I_1 + I_2 = I$; we have

$$H = \frac{R_1}{a_1}.I = \frac{R_2}{a_2}.I = M. \quad . \quad . \quad . \quad . \quad (26)$$

in either of which equations the value a_1 a_2 and I can be computed; while R_1 and R_2 are obtained by applying a suitable coefficient of safety to tabular ultimate resistances. These equations of safety indicate that H must not exceed either of the values equated with it.

Second when it is desirable that there should not be any waste of material in the section; then resistance of both sorts on the two sides of the neutral plane must be fully employed and the section will be one 'of equal strength.' Let r_1, r_2 be the strains actually induced by H corresponding to but less than R_1, R_2, the safe strains that might be induced without detriment; the case will then require that—

$$r_1 : r_2 : r_1 + r_2 :: a_1 : a_2 : a_1 + a_2. \quad . \quad . \quad . \quad (27)$$

that is, the position of the neutral plane must be such that the whole depth of the section is *divided* by it into two parts proportional to the actual strain intensities. This necessitates the employment of some special form of cross-section, as a **T** section or an **I** section.

The preceding cases apply merely to isotropic material.

Third.—With material that is not isotropic, as for instance with cast iron, the tensile resistance of which is comparatively small, sections of very unequal flanges are employed in girders, with the larger flange so placed as to resist tension. But the exact position of the neutral plane of any such girder cannot be determined.

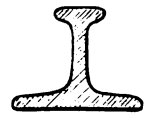


FIGURE 32.

If the assumption be made that the section adopted is one 'of equal strength,' the whole depth of section is divided into two parts having the same ratio as the two presumed resistances of material, and then a neutral axis is assumed; otherwise, the neutral axis might be anywhere within the effective depth in flanged girders. The limitation to effective depth or even to clear depth between flanges, is based on the practically correct assumption that the two flanges do resist the two stresses, or that the centres of stress and strain are within the flanges. This is equivalent to saying that the section is already of nearly suitable form, leaving its dimensions variable at will.

It is usual, therefore, first to assume that the neutral plane is at the centre of gravity at every cross-section, and to apply then the equation of safety,

$$H=\frac{R_1 I_1}{a_1}+\frac{R_2 I_2}{a_2}$$

as before used at page 96; and afterwards to assume that the neutral plane shifts to a certain extent to either side of that position, and reapply the equation in those two cases. The amount of total shift is empirical, differing according to various practice, some persons using $\frac{1}{4}$ and some $\frac{1}{2}$ and more of the effective depth, while others ignore shift and imagine the neutral plane to be positively through the centre of gravity. The reason for this last assumption

does not, however, rest on demonstration, as far as can now be discovered.

Beams of 'uniform section' and beams of 'uniform strength' are so far similar that the safe equations of stress and induced strains must hold at every section throughout their length. But as the horizontal stress varies under lateral load (see chapter ii. page 18), the beam of uniform section will have a section of greatest stress, for which, if the safe equation holds, the rest of the beam will necessarily have an excess of material. Economy reduces all such excess, in a beam or girder of uniform strength or of varying section, by so disposing the material at every section that the special stress occurring there is counteracted by merely sufficient safe strain; so that the equation of safety varies at every section in accordance with varied values of H. The beam of uniform strength may, therefore, vary either in depth or in thickness, or in both respects, as may be preferred, so as to fulfil this condition.

Alternative Method.—Another mode of representing the moment of strain at a section is to adopt the simple form

$$M = M_1 d' = M_2 d' \quad . \quad . \quad . \quad . \quad . \quad (28)$$

for the top and bottom flanges of a deep girder;
where d' = effective depth of girder,
= depth of web + half sum of flange-depths;
but this approximative method is merely applicable in certain cases of fully investigated section.

Deflexion under resolved transverse stress.

The elastic deflexion, or deflexion important in engineering solutions, is the bend, produced in a beam or girder by stress from transverse load, from which there is

perfect recovery on discontinuance of stress. A set-deflexion, permanent deflexion, or sagging, remaining after the discontinuance of a transverse load or stress, is an entirely separate matter, not to be confounded with it. The difference is usually accounted for by saying that the former deflexion is produced within the elastic limit, and that the latter is not; but this statement is not absolutely correct, though it is nearly true generally.

A small permanent sag may result from the first application of a stress well within elastic limits, or less than proof-stress, as usually termed.

The elastic deflexion, or recoverable bend, which, with a girder or column under any stress that may be concentrated representatively at midspan, takes a curve termed 'the elastic curve,' affords evidence of flexibility, but not necessarily either of impaired strength of the structure bent, or of weakness of the sort of material used; stiffness and strength require independent demonstration.

The flexure (ξ) of a beam or column, represented by the greatest deflexion at midspan when under stress, is the most useful practical result obtained from the determination of the elastic curve, as it is a means both of comparing deflexions of beams, &c., of different lengths of the same material, and of those of beams of different material of the same length. Theoretically, however, and for purposes of engineering solution, the various formulæ determining the whole of the properties of the elastic curve are more valuable, as they sometimes aid in determining values of the stresses that produce the curvature; hence their special importance.

Treating the transverse load or stress on a beam or column as resolved into lateral stresses V inducing shearing strain, and lengthway stresses H (see chapter ii.); the con-

sideration of the former may be entirely neglected as regards deflexion, while the values of H at every section throughout the length are the stresses or moments of stresses producing deflexion; these are hence often termed bending stresses or bending moments; with horizontal beams they are also horizontal stresses.

Deflexion of a horizontal beam.—A horizontal beam is the simplest case for deducing the elastic curve, as the weight of the beam itself acts equally all throughout the curve in augmenting deflexion.

To obtain the radius of curvature of the neutral axis, at any section in a strained horizontal beam, we have three conditions.

1. As shown under Transverse Strain in the last paragraph, we have

$$H=\frac{R_1 I}{a_1},$$

where a_1 is the axial distance of the remotest lamina in the tension prism.

2. We have the equation for tensile stretch or elongation (l_1) of a unit of length (l) taken at this distance from the neutral axis when the beam is bent, which is

$$\frac{l_1}{l}=\frac{R_1}{E_1},$$

where E_1 is the elasticity after tension. See page 84.

3. From geometrical consideration of the curvature produced, and the similar triangles, whose corresponding sides are ρ the radius of curvature, l the length unit originally on the neutral axis in the large triangle, and in the small triangle a_1 the axial distance, l_1 the stretch, we have

$$\frac{\rho}{l}=\frac{a_1}{l_1}.$$

Combining these three equations we have

$$\rho = \frac{E_1 I}{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Also if we deal with the compressed prism we should similarly obtain

$$\rho = \frac{E_2 I}{H}.$$

Under the conditions of isotropic material these are equal; also the existence of isotropy was assumed in the equation of condition No. 1; hence generally equation (29) holds with isotropic material only as a mode of obtaining the radius of curvature.

With materials of other sorts, as cast iron, the same principles may be applied in equations of rather similar form, in order to obtain the stress H, that will sustain an already produced curvature of a given radius ρ, when the elasticities are given and the section is specially formed to suit the case; but the converse is an indeterminate problem.

Returning to isotropic material, the value of the radius being now obtainable at every section, the deflexion curvature may be drawn, and all further details connected with it may be obtained graphically. But the great length of radius and the smallness of the ordinates and of the inclinations at every section render this practically inconvenient in many cases.

Proceeding, therefore, to determine ordinates and inclinations analytically on the assumption that they are small, we must employ the differential equation of the radius of curvature ρ, when δx is taken along the axis of abscissæ, δy along ordinates; and the origin is taken at the point

where the curve either becomes horizontal or ceases to curve. This is

$$\rho = -\{1 + (\partial_x y)^2\}^{\frac{3}{2}} \div \partial_x^2 y,$$

in which $(\partial_x y)^2$ may be rejected, being the square of an already very small quantity; so that

$$\rho = -\partial_y^2 x = \frac{EI}{H}; \quad . \quad . \quad . \quad . \quad . \quad (30)$$

or

$$\partial_x^2 y = -\frac{H}{EI}.$$

With a free horizontal beam under uniform transverse load the elastic curve becomes horizontal at midspan, a point convenient as an origin of co-ordinates. Taking it so, the value of x will then be zero, and that of y will also be zero; so also $\partial_x y =$ tan α, α being the inclination to the horizon will be zero; at the end of the beam where $x = \frac{1}{2}l$, y will be the greatest; that is, $y = \xi$ the flexure, and $\partial_x y$ $=$ tan α will there give the greatest inclination in the curve.

Also at any section between these two extreme values of x we have

$$\partial_x y = \tan \alpha = {}_0\!\int^{\frac{1}{2}l} \frac{H}{EI} \cdot \partial x \quad . \quad . \quad . \quad . \quad (31)$$

$$y = {}_0\!\int^{x}_{x} {}_0\!\int^{x}_{x} \frac{H}{EI} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Also when $x = \frac{1}{2}l$; $\quad y = \xi = {}_0\!\int^{\frac{1}{2}l}{}^{2}_{x} \frac{H}{EI}. \quad . \quad . \quad . \quad . \quad (33)$

formulæ giving the ordinates and inclinations at all points, when H, E, and I are given and the material is isotropic.

Conversely also, we may by the help of such formulæ sometimes obtain unknown values of H in various cases of beams deflected under certain given conditions, which afford values of tan α at the points of support or points of contrary flexure, giving additional values of x and y. Such Engineering Solutions will be treated under their own special cases.

Deflexion of a Cantilever.—With a cantilever having a representative transverse stress or load at its free end, and a length l, the principle afforded by the general equation (30)

$$\partial_x^2 y = -\frac{H}{EI}$$

also holds good. But in this case the curvature is horizontal at the fixed end, which thus becomes a convenient origin of co-ordinates x and y; and there

$$x=0; \qquad y=0; \qquad \partial_y x = \tan \alpha = 0;$$

hence at the free end $x=l$; $y=\xi$ the flexure; $\partial_y x = \tan \alpha$ is then the greatest inclination in the curve; and the three equations corresponding to those of (31), (32), and (33) are for all intermediate sections

$$\partial_x y = \tan \alpha = {}_0^l\!\int \frac{H}{EI} \partial x \quad . \quad . \quad . \quad . \quad (34)$$

$$y = {}_0^x\!\int_x^2 \frac{H}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Also when $x=l$;

$$y=\xi={}_0^l\!\int_x^2 \frac{H}{EI} \quad . \quad . \quad . \quad . \quad . \quad (36)$$

the conditions being otherwise similar to those before-mentioned as regards given terms and isotropy of material.

Other special cases will be treated as Engineering Solutions.

Deflexion in a fixed curved rib of uniform section and circular curvature.—The formulæ through which the deflexions may be arrived at in representative terms are thus obtained—

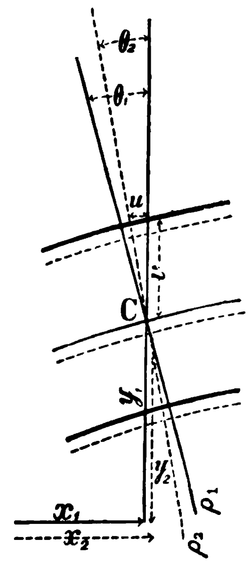


FIGURE 33.

Let ρ_1 ρ_2 be the radii of curvature before and after deflexion respectively;

Let θ_1 θ_2 be the inclinations to verticality of any section, taken normally to the original curvature, before and after deflexion;

$x_1 y_1$ the rectangular co-ordinates, from an origin at the end of the rib, of a point C on a neutral axis, at this section before deflexion;

x_2 y_2 the co-ordinates of the same point after deflexion;

then $x' = x_2 - x_1$ is the horizontal displacement;

and $y' = y_1 - y_2$ is the deflexion vertically;

also let E be the modulus of elasticity,

and s the total length of arc of the curved rib.

Now, dealing with the strained prism at the given section (see figure 33), let u and v be the horizontal and vertical co-ordinates from C, in which the strained laminæ may be expressed.

Then δu, δv, δs are corresponding infinitesimal elements, or differentials of those three terms; and as the radius of curvature is large in comparison with the half-depth of rib, we may when dealing with limiting ratios, &c., of infinitesimals, assume the condition to hold that practically amounts

to saying that the small horizontal ordinates are tangential to the curve, and the small vertical ordinates are normal to it. This assumption, of course, could not apply in small ribs of sharp curvature and of great depth.

Now, as the length of the curved rib is practically unaffected by deflexion in small parts of it,

$$\partial s_1 = \partial s_2 = -\rho_1 \partial \theta_1 = -\rho_2 \partial \theta_2 = \partial x_1 \sec \theta_1 = \partial y_1 \operatorname{cosec} \theta_1 = \partial x_2 \sec \theta_2 = \partial y_2 \operatorname{cosec} \theta_2.$$

Also the length of any fibre strained within the prism is before deflexion $\partial s + v\partial\theta_1$, and after deflexion is $\partial s + v\partial\theta_2$; while the sectional area of the fibre is $\partial u.\partial v$, and $E\partial u\partial v$ will be the force necessary to elongate $\partial s + v\partial\theta_1$ to an extent or stretch equal to its original length.

By the principle of elongation of fibre under strain (see Tension, page 84), the force necessary to elongate the same fibre by a stretch or length $(\partial\theta_2 - \partial\theta_1)v$ will be equal to

$$E.\partial u\partial v.\frac{(\partial\theta_2 - \partial\theta_1).v}{\partial s + v\partial\theta_1} = E.\partial u.\partial v.\frac{(\partial\theta_2 - \partial\theta_1)v}{(\rho + v).\partial\theta_1};$$

treating ρ as a mean radius of curvature.

Also the moment of this force about the neutral axis is this force multiplied by v; hence, the total moment of the complete section when symmetrical about the neutral axis is

$$M = \frac{E.(\partial\theta_2 - \partial\theta_1)}{\partial\theta_1}.\iint \frac{2\partial u.\partial v.v^2}{\rho\left(1 - \frac{v^2}{\rho^2}\right)} = \frac{EI}{\rho}.\frac{(\partial\theta_2 - \partial\theta_1)}{\partial\theta_1};$$

when $\frac{v^2}{\rho^2}$ may be treated as inconsiderable.

Hence

$$\theta_2 - \theta_1 = \frac{\rho^2}{EI}\int \frac{M}{\rho}.\partial\theta_1;$$

but
$$\theta_2-\theta_1=+\frac{\partial x_2-\partial x_1}{\rho\sin\theta_1.\partial\theta_1}=-\frac{\partial y_2-\partial y_1}{\rho\cos\theta_1.\partial\theta_1};$$

$$\therefore\; x'=x_2-x_1=\frac{\rho^3}{EI}\int\left[\sin\theta_1.\partial\theta_1\int\frac{M}{\rho}.\,\partial\theta_1\right]; \qquad (37)$$

and
$$y'=y_2-y_1=\frac{\rho^3}{EI}\int\left[\cos\theta_1\partial.\theta_1\int\frac{M}{\rho}\partial\theta_1\right]; \qquad (38)$$

the required representative values.

Strain in a Bridge-pier or Abutment.

In Chapter II. the external forces acting on a bridge-abutment and on a bridge-pier have been treated; the position of the point of application of resultant stresses or thrust at any horizontal course was also determined by co-ordinates. It is now required to determine the strains per unit of section induced by the resultant stress.

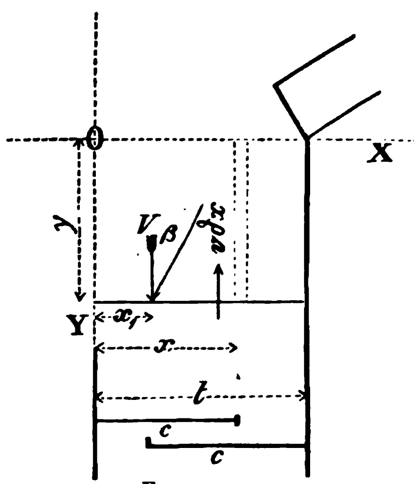


FIGURE 34.

Taking as in the figure a section of pier on the centre line; OX, OY rectangular axes, and dealing with a horizontal course of masonry or brickwork at a depth y below the axis OX; let V a vertical component of resultant stress or thrust, act at a distance x_1 from the axis OY; and let t be the thickness of the abutment, or pier under unbalanced lateral forces.

Also let $v\partial x$ acting at x be an elementary portion of the resistance offered by the material all over this horizontal course at the depth y.

Equating the stress and strain

$$V={}_0\int^t v\partial x\text{; and } Vx_1={}_0\int^t vx\partial x.$$

As the material may be treated as rigid, the distribution of strain will follow a straight line, so that $v=ax+b$; whence

$$V=\tfrac{1}{2}at^2+bt\text{; and } Vx_1=\tfrac{1}{3}at^3+\tfrac{1}{2}bt^2\text{;}$$

$$a=-\frac{6V}{t^3}(t-2x_1)\text{; and } b=+\frac{2V}{t^2}(2t-3x_1)\text{;}$$

$$\therefore\ v=-\frac{6V}{t^3}\cdot(t-2x_1)x+\frac{2V}{t^2}(2t-3x_1).\quad .\quad .\quad (39)$$

But as tensile strain is inadmissible, V must not be negative; considering, therefore, its values at the two limits, that is, when $x=0$ and $x=t$; we find that

$$v_0=\frac{2V}{t^2}(2t-3x_1)\text{; and } v_t=\frac{2V}{t^2}(3x_1-t),$$

whence to prevent v from becoming negative,

x_1 must be less than $\frac{2}{3}t$ and greater than $\frac{1}{3}t$ . . (40)

a deduction that will either limit the position of V, or alternatively assert a definite length of horizontal course that will alone be effective in offering resistance to it.

Let c=that definite length,
then if x_1 be less than $\frac{1}{3}t$,

$$V={}_0\int^c v\partial x=\tfrac{1}{2}v_0c\text{; and } Vx_1={}_0\int^c vx\partial x=\tfrac{1}{6}v_0c^2\text{;}$$

also if x_1 be greater than $\frac{2}{3}t$,

$$V={}_{t-c}\int^t v\partial x=\tfrac{1}{2}v_tc\text{; and } Vx_1={}_{t-c}\int^t vx\partial x=\tfrac{1}{2}v_tc(t-\tfrac{1}{3}c)\text{;}$$

whence in these two cases we have, when x_1 is less than $\frac{1}{3}t$,

$$c = 3x_1 \text{; and } v_0 = \frac{2V}{3x_1} \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

and when x_1 is greater than $\frac{2}{3}t$,

$$c = 3t - 3x_1 \text{; and } v_t = \frac{2V}{3t - 3x_1} \quad . \quad . \quad . \quad . \qquad (42)$$

The vertical strain on the section is thus evaluated; the same units of weight being used in both V and v, and the same linear units throughout, so that $v \div b$ is the strain per unit of surface corresponding to the linear unit, b being the breadth of section.

Thus, if feet be used throughout, $\frac{v}{b}$ is the strain per square foot.

Also the strain per unit of section in the direction of the resultant is $\frac{v}{b} \sec \beta$.

Strains in Tubes under pressure.

Cylindrical tubes may be intended to sustain either internal pressure or external pressure; in both cases purely theoretic demonstration fails to arrive at a just thickness of tube corresponding to a given pressure and given radius of section.

Practically the method of direct experiment on several samples, by causing them to burst or to collapse under pressure, and applying a coefficient of safety to the pressure of destruction, is the only trustworthy mode of arriving at a safe pressure for the tubes or material used by the maker; but even these vary at different times. The coefficient for slow moving loads, which is $\frac{3}{16}$ for cast iron (see Table) is suitable to water-pressure in distribution,

but $\frac{1}{6}$ is most commonly used. Also, with respect to water-pipes of empirical standard thicknesses, it is usual to assume that if they do not succumb to treatment in manufacture and shocks in laying, they will remain uninjured afterwards under ordinary treatment.

Apart from known or experimental bursting pressures, theoretical equations of stress and strain, or equations between theoretical pressure and known tabular resistance of material, tensile, compressive, &c., fail to give any solution of value.

Some of the methods tried may, however, be indicated.

First with internal pressure; one method was to treat the ring-thickness as bar-thickness equally subject to tension; another to allow tension decreasing with the square of the distance from the centre, and to integrate the strains, as total tension. This is Barlow's method, and yields—

$$t=\frac{\rho P}{R-P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

where R is the ultimate resistance of material, and P the pressure both in similar units, t is the thickness and ρ the internal radius, both in similar units.

Secondly, with external pressure. Fairbairn adopted a special coefficient of ultimate resistance to collapse of 9 672 000 pounds to the square inch for plate iron flues, and applied it in the formula

$$R=9\,672\,000\,\frac{t^2}{ld} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

where R=ultimate resistance in lbs. per square inch,

l, d, and t, are the length, diameter, and thickness, in any similar units.

With elliptic sections, the above equation is modified into

$$R = 9\,672\,000 \,.\, \frac{t^2 . b}{2la^2};$$

where a and b are the semi-axes, major and minor, of the elliptic section; and the units are similar as before.

Both of these methods failing, empirical rules of various sorts have been employed, such as those mentioned in the Solutions hereafter.

Natural Expansion.—The three sources of natural expansion of material used in engineering structures are heat or increase of temperature, moisture, which fills up the pores and swells a mass, and frost, which expands the moisture present in a mass of material, and thus aids in molecular disintegration. All these expansions are accompanied by action in the form of stress as well as of potential energy, although their values cannot be always estimated. The mere extension or stretch of material due to heat is given in the tables following this chapter for a few substances. Such expansion is estimated for dry materials. Although moisture has an evident effect in swelling many materials, yet with timber, experiment has shown that when it occurs coincidently with heat, it often has the effect of reducing the heat-expansion. The absorption of which some few materials are capable is also given in the Tables.

The effect of heat on materials extends beyond mere elongation, in some cases it affects their strength; but the investigation of such matters has hardly been yet carried out by physicists to the extent of yielding generally useful results. The strains induced by expansion stresses follow the same laws as other strains.

NOTATION ADOPTED IN STRAINS, &c.

An ultimate resistance or modulus of strength being unused generally in the formulæ is not symbolised.

A coefficient of safety applied to ultimate resistances for obtaining safe resistance is hence also not symbolised.

Rr, R_1r, &c.	Moments of resistances or of strains.
R, R_1, R_2, &c.	Safe resistances, or safe strains per unit of sectional area.
r, r_1, r_2, &c.	Actual resistances induced by stress, or actual strain, per unit of sectional area.
E, E_1, E_2, &c.	Moduli of elasticity per unit of sectional area.
S, S_1, S_2, &c.	Sectional areas.
I, I_1, I_2, &c.	Moments of inertia of sections.
M, M_1, M_2, &c.	Moments of resistance of sections.
l, l_1, &c.	Lengths.
b, b_1, &c.	Breadths.
d, d_1, &c.	Depths.
a, a_1, a_2, &c.	Extreme axial distances, or distance of furthest lamina from an axis of rotation.
ξ	Flexure, or greatest deflection.
x, y, z	Co-ordinates of length, breadth, and depth, respectively, when all are used together.
dx, dy, dz	Differentials of x, of y, of z.
$\frac{dx}{dy}$, $\frac{d^2x}{dy^2}$, &c.	First, second, &c., differential coefficients of x with regard to y.

α	inclination to the horizon.
β	inclination to the vertical.
U, u, &c.	work performed in resilience.
y, y_1, &c.	deflexions of a horizontal beam.
ρ, ρ_1, &c.	radii of elastic curves, &c.
$\int_y X$; $\int_y^2 X$; or $\int X \partial y$; $\int^2 X \partial^2 y$;	first and second, integrals of X with regard to y.
${}^c_e\!\int_y X$; ${}^c_e\!\int_y^2 X$; or ${}^c_e\!\int_y X \partial y$; ${}^c_e\!\int_y^2 X \partial^2 y$;	definite integrals with regard to y, within limits c and e.

Whenever it is not otherwise expressed, the units adopted in the terms of any formula are of the same or of corresponding sorts throughout it, so that these may be chosen at will.

Symbols denoting the natures of resistances and elasticities are seldom required, as the reader knows from context and judgment the natures of these, whenever they are used.

Table of Moments of Resistance of Sections symmetrical about an Axis through the Centre of Gravity.

	S_1 or S_2	a_1 or a_2	c_1 or c_2	g_1 or g_2	I_1 or I_2	I	M
Rectangle, with depth vertical d=depth . . . . b=breadth . . .	$\frac{1}{2} b d$	$\frac{1}{2} d$	$\frac{1}{3} d$	$\frac{1}{4} d$	—	$\frac{1}{12} b d^3$	$\frac{1}{6} R b d^2$
Square, with diagonal vertical h=height or diagonal .	$\frac{1}{4} h^2$	$\frac{1}{2} h$	$\frac{1}{4} h$	$\frac{1}{6} h$	—	$\frac{1}{48} h^4$	$\frac{1}{24} R h^3$
Circle . . . . . . d=height or diameter .	$\frac{1}{8} \pi d^2$	$\frac{1}{2} d$	$\frac{3}{32} \pi d$	$\frac{2}{3\pi} . d$	—	$\frac{1}{64} \pi d^4$	$\frac{1}{32} \pi R d^3$
Pair of equal isosceles triangles . . . . . d=sum of two altitudes vertical . . . . b=base of either triangle .	$\frac{1}{4} b d$	$\frac{1}{2} d$	$\frac{3}{8} d$	$\frac{1}{3} d$	—	$\frac{1}{16} b d^3$	$\frac{1}{8} R b d^2$
Ellipse, major axis vertical . h=height, major . . b=breadth, minor . .	$\frac{1}{8} \pi b h$	$\frac{1}{2} h$	$\frac{3}{32} \pi h$	$\frac{2}{3\pi} h .$	—	$\frac{1}{64} \pi b h^3$	$\frac{1}{32} \pi R b h^2$
Symmetrical Isotropic Section generally	$S_1 = S_2$	$a_1 = a_2$	$c_1 = c_2$	$g_1 = g_2$	$I_1 = I_2$	$I_1 + I_2$	$\frac{R_1 I_1}{a_1} + \frac{R_2 I_2}{a_2}$

Moments for complicated forms are obtained numerically with a mechanical integrator.

Tables of Qualities of Materials.

Coefficients of Safety.

To be applied to ultimate resistances of materials in order to obtain safe resistances.

1. Under steady, gradually applied load or stress.

	Coefficients.
Strongest steel	$\frac{1}{2}$
Ordinary steel	$\frac{1}{3}$
Wrought iron	$\frac{1}{3}$
Cast iron	$\frac{1}{4}$
Brickwork and masonry—good	$\frac{1}{4}$
" " inferior	$\frac{1}{5}$
Timber, seasoned; good workmanship	$\frac{1}{4}$
" inferior quality and workmanship	$\frac{1}{5}$
Cordage, new and untarred	$\frac{1}{4}$

2. Under sudden stress, or moving load liable to cause sudden stress.
The tabular coefficients are respectively halved.

3. Under slow-moving loads.
Use values three-quarters of the tabular coefficients.

4. Under mixed loads, partly sudden.
Double the value of any sudden load or stress, increase slow-moving loads by a half, and add them to the steady stress, then treat the sum as steady stress representatively.

Proof-strength as Safe Resistance.

The proof-resistances of materials can only be obtained by experiment; approximations from tables of experimental values on corresponding material are useful as approximate substitutes; but coefficients for calculating proof-resistances from ultimate resistances are generally useless, as the results are merely nominal, not actual proof-resistances. The usual factor of proof-resistance for all sorts of iron and steel is $\frac{1}{3}$. Materials may be safely employed in structures, &c., in certain cases up to their full proof-strength. These cases are, when only one sort of strain can possibly happen, when much variation in strain does not occur, and when a very slight set would not be productive of danger.

The commonest condition is, therefore, that of solid continuous bodies under compressive strain of tolerably even amount. Bodies liable to any alternating strains require the coefficients of safety before mentioned.

Moduli of Elasticity and of Resilience.

(Selected from Moseley's and other Collections.)

	Tensile Elasticity	Shearing Elasticity	Tensile Resilience	Expansion 0° to 100° Celsius
TIMBER:—				
Acacia	1 152 000	—	—	—
Ash	1 644 800	76 000	—	0·0005
Beech	1 353 600	—	—	—
Birch, English	1 562 400	—	—	—
Birch, American	1 257 600	—	—	—
Elm	699 840	76 000	—	—
Fir, Riga	1 099 200	—	—	—
Larch, green	897 600	—	—	—
„ dry	1 052 800	—	—	0·0004
Oak, English	1 451 200	84 000	—	—
„ Canadian	2 148 800	—	—	—
Pine, pitch	1 225 600	—	—	—
„ red	1 840 000	116 000	—	—
Teak, dry	2 414 000	—	—	—
METALS:—				
Brass cast	8 930 000	—	—	0·0019
„ wire	14 230 000	5 330 000	—	0·0021
Copper wire	17 000 000	6 200 000	—	0·0017
Gun-metal, 8 to 1	9 900 000	—	—	0·0018
Cast-iron (hot blast):				
Carron, No. 2	16 085 000			
Carron, No. 3	17 873 000	av.	13	
Devon, No. 3	22 473 500	2 850 000	to	0·0011
Buffery, No. 1	13 730 500		37	
Coed Talon, No. 2	14 320 500		av. 18	—
Wrought Iron: Plate	24 000 000	8 500 000	104	0·0012
„ Bar	29 000 000	to	124	—
„ Wire	25 300 000	9 500 000	320	0·0014
„ Wire rope	15 000 000		—	—
Lead, sheet	720 000	—	—	0·0028
Steel: Puddled	20 000 000 to 25 000 000	—	279	—
„ Bars	30 000 000 to 40 000 000	—	415	0·0011
„ Hard tempered	42 000 000	—	—	—
Tin cast	4 608 000	—	—	0·0021
Zinc	13 680 000	—	—	0·0029

Moduli of Elasticity and of Resilience—continued.

	Tensile Elasticity	Shearing Elasticity	Tensile Re-silience	Expansion 0° to 100° Celsius
VARIOUS MATERIALS:—				
Glass plate . . .	8 000 000	—	—	0·0009
Marble . . .	2 520 000	—	—	0·0009
Slate, Welsh . .	15 800 000	—	—	0·0010
„ Westmoreland .	12 900 000	—	—	—
„ Scotch . .	15 700 000	—	—	—
Granite . . .	—	—	—	0·0009
Sandstone . . .	—	—	—	0·0011
Brick	—	—	—	0·0036
Cement . . .	—	—	—	0·0014

Moduli of Ultimate Strength in lbs. per sq. inch

(according to Barlow, Bevan, Muschenbroek, Hodgkinson, and Tredgold).

(Selected from Moseley's Collection.)

TIMBER	Specific gravity	*T* Tenacity	*C* Crushing	*S* Shearing	*B* Cross-breaking
Acacia, English . .	0·710	16 000	—	—	11 202
Alder	0·800	14 186	6 895	—	—
Ash	0·767	17 207	9 023	1 400	12 156
Bay-tree . . .	0·882	12 396	7 158	—	—
Beech	0·772	16 817	8 548	—	9 336
Birch, English . .	0·792	15 000	5 466	—	10 920
„ American .	0·648	—	9 033	—	9 624
Box	0·960	19 891	19 299	—	—
Cedar, fresh . .	0·909	11 400	5 674	—	7 400
Elder	0·695	10 230	8 467	—	—
Elm, seasoned . .	0·588	13 489	10 331	1 400	6 078
Fir, Riga . . .	0·753	12 203	6 667	700	7 110
Hornbeam . . .	0·760	20 240	7 289	—	—
Larch, green . .	0·522	10 220	3 201	970	4 992
„ dry . .	0·560	8 900	5 568	1 700	6 894
Mahogany, Spanish .	0·800	16 500	8 198	—	7 600
Oak, English . .	0·934	17 300	7 097	2 300	10 032
„ Canadian . .	0·872	10 253	6 870	—	10 596
Pine, pitch . .	0·660	7 818	—	—	9 792
„ red . . .	0·657	—	5 375	800	8 946
Poplar	0·383	7 200	4 115	—	—
Sycamore . . .	0·690	13 000	—	—	9 600
Teak	0·657	15 000	12 101	—	14 772
Walnut	0·671	8 130	6 645	—	—

Moduli of Ultimate Strength in lbs. per sq. inch

(according to Fairbairn, Hodgkinson, Tredgold, Rennie, and Muschenbroek).

METALS	Specific gravity	*T* Tenacity	*C* Crushing	*S* Shearing	*B* Cross-breaking
Aluminium bronze	7·680	73 000	132 000	—	—
Antimony, cast	4·5	1 066	—	—	—
Bismuth, cast	9·81	3 250	—	—	—
Brass, cast	8·40	17 968	10 304	—	—
„ wiredrawn	8·54	—	—	—	—
Copper, cast	8·61	19 072	—	—	—
„ sheet	8·79	—	—	—	—
„ wiredrawn	8·88	61 228	—	—	—
„ in bolts	—	48 000	—	—	—
Iron, wrt., in plate	7·70	50 000	—	av. 50 000	44 000
„ „ in bars	7·70	60 000	—		
„ „ hammered	—	70 000	—	—	—
„ „ in wire o·1 'dr.	- -	90 000	—	—	—
Iron, cast (hot blast)					
„ Carron No. 2	7·046	13 505	108 540		37 503
„ Carron No. 3	7·056	17 755	133 440		42 120
„ Devon No. 3	7·229	29 107	145 435	av. 27 700	43 497
„ Buffery No. 1	6·998	13 434	86 397		35 316
„ Coed Talon No. 2	6·968	16 676	82 739		33 145
Lead, cast	11·45	1 824	—	—	—
„ sheet, milled	11·41	3 328	—	—	—
Steel, soft, in bars	7·780	120 000	—	—	—
„ „ in plates	—	av. 100 000	—	—	—
„ hard, tempered	7·840	140 000	-	—	—
„ cast	-	—	260 000	—	—
Tin, cast	7·291	5 322	—	—	—
Zinc	7·028	—	—	—	—

Moduli of Ultimate Strength in lbs. per sq. inch.

Various Materials	Specific gravity	*T* Tensile	*C* Crushing	*S* Shearing	*B* Cross-breaking
Asphalte . . .	2·500	—	—	—	—
Brick, strong . .	2·168	280	800 to 1 000	—	—
„ weak . .	2·085	300	560 to 800	—	—
Basalt and syenite .	—	—	12 000	—	—
Brickwork . . .	1·800	See Mortars		—	—
Cement mortar . .	—	50 to 400	1 500 to 3 000	—	—
Cement, Portland .	1·800	—	—	—	—
Concrete . . .	2·0 to 2·3	—	—	—	—
Glass, plate . .	2·453	9 420	30 000	—	—
Granite, Aberdeen .	2·526	—	10 000	—	—
„ Cornish. .	2·662	—	13 000	—	—
Hemp-cable . .	—	5 600	—	—	—
Marble, Italian . .	2·638	—	5 000	—	1 062
„ Galway . .	2·695	—		—	2 664
Lime mortar . .	1·751	50	—	—	—
Limestone, weak .	2·700	—	4 000	—	—
„ strong .	—	—	8 500	—	—
Sandstone, weak . .	2·300	—	3 500	—	1 700
„ strong Yorkshire .	—	—	9 800	—	—
Slate, Welsh . .	2·888	12 800	—	—	11 766
„ Westmoreland .	2·791	—	—	—	..
„ Valentia . .	2·880	—	—	—	5 226
„ Scotch . .	—	9 600	—	—	.
Tile	1·815	—	—	—	—

Average Heaviness of Earths, Rocks, &c.

	Specific gravity.
Basalt	2·5 to 2·9
Chalk	1·8 to 2·5
Clay	1·9 to 2·1
Coal anthracite	1·70
„ bituminous	1·33
„ cannel	1·25
Earth	1·5 to 2·0
„ loose	1·50
„ rammed	1·584
Felspar	2·60
Flint	2·63
Freestone	1·90
Gravel	1·75 to 1·95
Granite	2·50 to 2·80
Gypsum	2·25
Loam	2·05
Marl	1·6 to 1·8

	Specific gravity.
Mud	1·7 to 1·9
Limestones	2·5 to 2·9
Marble	2·6 to 2·7
Oolite	2·0 to 2·5
Peat	1·25 to 1·50
Pozzolano	2·68
Quartz	2·65
Sand, damp	1·89
„ dry	1·50
Sandstones	2·2 to 2·8
Shingle	1·35 to 1·50
Shale	2·2 to 2·5
Serpentine	2·60
Syenite	2·62
Slate, av.	2·7 to 2·9
Trap	2·70
Whinstone	2·4 to 2·9

Angles of Repose.

	Angle	Slope to Unity
Clay, damp	45°	1·00
„ wet	14° to 17°	4·00 to 3·23.
Sand and clay, mixed	21° to 37°	2·63 to 1·33
Gravel	35 to 48	0·90 to 1·43
Shingle	„ „	anything
Peat	14° to 45°	4· to 1·

Absorption of water possible.

	1 part in
Clay Slate	80 to 700
Granite	80 to 700
[illegible]	40
Yorkshire Sandstone	30 to 60
[illegible]	14 to 125

Tensile Strength of Pure Cement in lbs. per square inch.

Number of set of sample	Days of Immersion in Water				Days of Exposure to Air			
	7	30	60	90	7	30	60	90
1	498	740	569	683	427	569	512	540
2	412	540	540	796	526	626	839	910
3	569	768	839	939	612	697	768	853
4	740	811	939	1024	711	782	839	910
Tensile Strength, when 1 part Cement 2 parts Sand.								
1	185	242	284	327	213	327	341	526
2	256	327	398	455	284	412	498	569
3	284	327	356	384	327	412	526	569
4	356	398	441	469	299	370	455	569

Crushing Strength of Pure Cement in lbs. per square inch.

Number of set of sample	Days of Immersion in Water				Days of Exposure to Air			
	7	30	60	90	7	30	60	90
1	3755	5134	5305	5604	4210	5092	5177	5533
2	4054	4750	5518	5774	4423	5106	5703	6201
3	5106	5874	6770	7282	5376	6187	6884	7410
4	5334	5860	6614	7126	5191	5831	6542	6969
Crushing Strength, when 1 part Cement 2 parts Sand.								
1	2261	2290	2916	2958	1749	2432	3072	3143
2	1493	1707	2091	2389	1764	2176	2517	3072
3	1849	2418	2916	3527	2105	2917	3399	3755
4	2034	2745	3101	3485	2660	2987	3357	3641

The above is selected from the results of official tests made at Berlin on Portland cement, made at Stettin (1), Riga (2), and Alsen (3 and 4); published by Dr. Böhme in Dingler's 'Polytechnisches Journal.'

Each result was the average of five experiments. In tensile test, the sample had a breaking section of 0·775 square inch.

In compressive test the section crushed was 15·5 square inches; the dimensions of sample being 3·94 × 3·94 × 2·36 inches.

Tenacity of Wrought Iron and Steel, according to Experiments of R. Napier and Sons, conducted by Mr. Kirkaldy.

Wrought iron	lbs. per square inch
Angle iron, various	50 056 to 61 260
Bushelled iron from turnings	55 878
Durham plates	51 245
Lanarkshire bar	51 327 to 64 795
" plates	43 433 to 53 849
Lancashire bar	53 775 to 60 110
Russian bar	49 564 to 59 096
Scrap, hammered	53 420
" various	41 386 to 55 937
Staffordshire plates	46 404 to 56 996
" bar	56 715 to 62 231
Swedish bar	41 251 to 48 933
Yorkshire plates	52 000 to 58 487
" Lowmoor	60 075 to 66 390

Steel	lbs. per square inch
Cast steel bars, rolled and forged	92 015 to 132 909
" plates	75 594 to 96 280
Bessemer steel bars, rolled and forged	111 460
Blistered steel bars, rolled and forged	104 298
Homogeneous metal bars, rolled	90 647
" " " forged	89 724
" " plates, 1st quality	96 280
" " " 2nd "	72 408
Puddled steel bars, rolled and forged	62 768 to 71 484
" " plates	71 532 to 102 593
Shear steel bars, rolled and forged	118 468
Spring steel bars, hammered	72 529

The following are common averages of tenacity of wrought iron, &c.:—

Steel plates, 32 tons per square inch.
Wrought iron boiler plates, 25 tons.
„ „ bridge plates, 22 tons.
„ „ ship plates, 20 tons.
„ „ builder's plates, 16 tons.
Cast iron, in bar, 10 tons.

In Rectangular Plate.—Effect of Sectional Proportions in Reducing Strength and Elasticity of Metal. M. Barba.

METALS	Ratio of Width to Thickness	Limit of Elasticity in tons per sq. inch	Ultimate Strength in tons per sq. inch
Steel . . .	1·98	18·16	27·11
„ . . .	9·80	17·52	25·52
Copper . . .	1·53	5·33	15·24
„ . . .	7·60	5·08	14·70

Regulations about Steel and Wrought Iron.

English Admiralty rules fix the test limits for steel at 26 to 30 tons per square inch for plates, beams, and bars; Lloyd's rules fix them at 27 to 31 tons; Liverpool registry rules for 28 to 32 tons per square inch.

French Admiralty rules allow graduation of minimum strength in steel according to size and thickness of section; in plates from $\frac{1}{4}$ to $\frac{3}{4}$ inch thick $28\frac{1}{2}$ tons per square inch.

In the United States, wrought iron in girders may be worked to tensile strain of 10 000 lbs. per square inch; but in compression and shearing strain to 7 500 lbs. per square inch as a maximum; while in small ties of bracing, &c., a strain of only 6 000 lbs. per square inch is allowed.

English Admiralty rules for wrought iron plate in testing under tensile strain—

			per sq. inch	per sq. inch
1st Class	B. B.	Plate, lengthwise	22 tons, crosswise	18 tons
2nd „	B.	„ „	20 „ „	17 „

Also hot and cold forge-tests and angle-tests.

Lloyd's rules for wrought iron; test under longitudinal strain, 20 tons per square inch.

Liverpool Committee rules put the mean breaking strain at 20 tons per

square inch of the original section, and 24 per square inch of the broken section.

In England the Board of Trade rules that wrought iron in engineering structures, girders, &c., may not be strained to more than 5 tons per square inch, and steel to not more than $6\frac{1}{2}$ tons per square inch, unless special permission be granted; the conditions of permission are unknown. The Board of Trade regulations for safe strain on wrought iron in boilers are—

Drilled plates, lengthwise . . .	6 000 lbs. per square inch
„ „ crosswise . . .	5 375 „ „
Punched plates, lengthwise . .	5 000 „ „
„ „ crosswise . .	4 500 „ „

There are also regulations for joints and rivets.

The War Department tests for shields and armour-bolts are, tensile strains per square inch of original section for plate iron, lengthwise, 20 tons, crosswise 16 tons; angle iron lengthwise, 22 tons; rivet iron lengthwise, 23 tons.

The Indian Store tests are gradually applied tensile stresses in tons per square inch, different for five standard qualities of wrought iron, thus—

	I.	II.	III.	IV.	V.
Plates, lengthwise	24	23	22	21	20
„ crosswise	22	20	19	18	17
Round and square bars . .	27	26	25	24	23
Flat bars	26	25	24	23	22
Angle and T-iron . . .	25	24	23	22	21

The Belgian Classification of Steel, according to the Société Cockerill, Seraing.

Class	Tensile strength per square inch	Extension or stretch	Percentage of carbon
I	25 to 32 tons	20 to 27 per cent.	0·05 to 0·20 per cent.
II	32 to 38 „	15 to 20 „	0·20 to 0·35 „
III	38 to 40 „	15 to 20 „	0·35 to 0·50 „
IV	40 to [illegible] „	5 to 10 „	0·50 to 0·65 „

I. [illegible] Used in girder and boiler plate and [illegible] also in wire, nails and screws; it welds, but does [illegible]

II. [illegible] also in guns; it may weld, [illegible]

III. Hard steel. Used in springs, spindles, parts of machines, guide bars; also in rails and tires; it does not weld, but it does temper.

IV. Extra hard steel (or tool steel). Used in files, saws, tools, also in fine springs; it does not weld, but tempers very much

Engineers' Specified Tests of Materials.

These vary with requirements and conditions, for it is not necessary to use the best material in every case, but merely to apply the material used within its powers of resistance safe.

When the best materials are specified, certain fixed corresponding tests may be demanded; as, for instance, in the following cases of iron for bridge-building.

Best Staffordshire plates, tensile strength lengthwise 22 tons per square inch; elastic limit $7\frac{1}{2}$ tons per square inch, contraction of section at fracture 16 per cent.; forge test on plates when cold, bending to 80° with the grain, and to 40° across grain; shock test on bars and bolts, 1 ton falling through 25 feet; angle and T iron to bend when hot in any way, doubling up.

With cast iron, test bars $1'' \times 2''$ in section cast as samples of the castings must resist a collected weight of 1 ton placed transversely, the bearings 4 feet apart; surfaces and fractured sections to be free from defect.

Recent Articles on Materials.

The foregoing Tables constitute the ordinary stock of such matter, with a few additions, as used by Engineers in ordinary reference. Extended and minuter details require a volume or more specially devoted to Materials. For further reference as to information published since 1880, the following list of articles in 'Engineering' on Materials may be useful.

1880.

Akerman's Tests of Iron, &c.	pp. 77, 139, 159	January
On Riveted Joints	pp. 110, 128, 148, 194, 300, 350	Feb. &c.
Scott and others on Portland Cement	p. 413	May
Steel Wire	p. 481	June
Galbraith on Hardening Steel	p. 61	July
Barba on Testing Metals	p. 99	July
Davis on Compression of Fluid Steel	pp. 109, 120	August
Belgian Tests of Steel	p. 123	August
Kennedy's Tests of Boiler Plates	p. 619	December

1881.

Kollmann's Experiments on Iron and Steel .	p. 109	February
Ducley on Chemical Tests of worn Rails . .	p. 220	March
Manganese Bronze	pp. 229, 339	March
Kennedy's Tests of Riveted Joints	pp. 436, 459, 508, 588	April &c.
Richards' Plastic Metal	p. 513	May
Manganese Alloys	p. 543	May
Tensile Strength of Leather	p. 10	July
German Rail Test	p 128	July
Tests and Weight of Portland Cement . .	p. 659	December

1882.

Tests of Phosphor Bronze	pp. 133, 192	February
Matheson on Steel for Structures	p. 236	March
Richards' Tests of Mild Steel . . .	pp. 475, 502	May
Tests of Tripolith Mortar	p. 293	Sept.
Armstrong on Steel	p. 354	October
Pourcel on Strength of Steel Castings . .	p. 541	December
Snelus, Tests of Steel Rails .	pp. 605, 616, 621, &c.	December

1883.

Butler's Artificial Stone	p. 159	February
Tests of Dick's Delta Metal	p. 161	February
Test Tables of Annealed Steel Plates . . .	p. 593	June
Mann on Adhesion of Portland Cement . .	p. 153	August
Lang's Wire Rope	p. 537	December

1884.

Hackney on Test Pieces of Bars and Plates . . .		Feb. 15
Hall's Tensile Tests of Steel Crankshafts		April 18
On Submarine Concrete Blocks	p. 16	July
The Strongest Bronze	p. 17	July
Estimation of Manganese in Cast Iron . .	p. 347	October
Spectroscopic Examination of Iron . . .	p. 375	October

PART II.

ENGINEERING SOLUTIONS.

SECTION I.

HORIZONTAL GIRDERS AND CANTILEVERS. SEPARATE AND CONTINUOUS.

Free Girders supported at both ends.

The sections employed in such girders may be divided into two classes:

1. *Symmetrical sections*, that is, of form symmetrical about both axes of form, vertically and horizontally, such as the solid and hollow rectangle and square, the solid and hollow ellipse and circle, the **I** section, and forms used in wrought iron, and isotropic material.

2. *Unsymmetrical Sections*, that is, of form unsymmetrical about any horizontal axis of form, although laterally symmetrical about a vertical axis; such as the Hodgkinson, Vignoles, and trough sections, having unequal top and bottom flanges.

To prevent needless repetition of solutions with merely different sections, it will be presumed that the moment of

inertia I for any section is known. Some values of I are given in a table following Chapter III. on Strains.

Solution Number 1.—*Free girder of uniform and of symmetrical section of isotropic material, having an equally distributed load.*

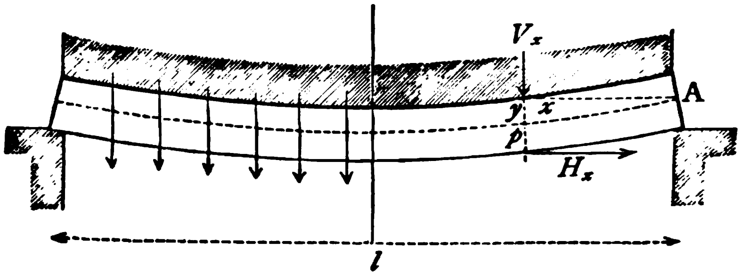


FIGURE 1.

1. The conditions of sufficient strength to safely resist horizontal stress.

Let ΣFf be the sum of the moments of external force with reference to any section of the girder;
- R be the safe resistance of material either to compression or tension;
- I the moment of inertia of the cross-section;
- a the extreme axial distance, or distance from the neutral axis to the furthest lamina of the section;
- l the span between points of support;
- w the load per unit of length of span.

In order that there may be sufficient strength throughout the girder, its weakest section must be safe, and at that section the equation of stress and safe strain at the ultimate will be expressed by

$$\Sigma Ff = H = M, \text{ moment of resistance.}$$

From Chapter II. on Stresses, pages 17, &c., and tabulated values on page 20, we have for the general expression for horizontal stress on a girder under loading of this kind

$$H_x = \tfrac{1}{2}wx\,(l-x),$$

where the variable x is measured from an origin at either support, as A in the figure. We have also under transverse strain, according to Chapter III. on Strains, p. 97, &c.

$$M = \frac{RI}{a}.$$

Hence the equation of safety at the ultimate is

$$\tfrac{1}{2}wx\,(l-x) = \frac{RI}{a}$$

and the weakest section must evidently occur at some point where either H is greatest or M is least. But as the terms involved in the value of M are constants, the weakest section is that where H_x is greatest; and when $\frac{1}{2}wx\,(l-x)$ is greatest, $x=\frac{1}{2}l$, and in that case $H=\frac{1}{8}wl^2$, and the weakest section is at midspan.

The equation of ultimate safety is then

$$w = \frac{8RI}{al^2}.$$

With all symmetrical sections of depth d, and breadth b, we have $a=\frac{1}{2}d$.

With the solid rectangle, $I=\frac{1}{12}bd^3$; and $w=\dfrac{4bd^2R}{3l^2}$.

With the solid ellipse, $I=\frac{1}{64}\pi bd^3$; and $w=\dfrac{\pi bd^2R}{4l^2}$.

The square and circle are merely particular forms of the rectangle and ellipse, where $b=d$.

With the hollow rectangle, $I=\frac{1}{12}(bd^3-b_1d_1^3)$;
where b_1, d_1 are the breadth and depth of the hollow;
and

$$w=\frac{4(bd^3-b_1d_1^3)R}{3l^2d}.$$

With the hollow ellipse, $I=\frac{1}{64}\pi(bd^3-b_1d_1^3)$,

and
$$w=\frac{\pi(bd^3-b_1d_1^3)R}{4l^2d}.$$

With the I section; this may be treated as a hollow rectangle, having the depth of web $=d_1$ and the sum of the breadths of the two side voids $=b_1$.

With other sections the values of I and a must be evaluated and applied in the general formula.

2. Conditions of sufficient strength to safely resist vertical stress :—

The general expression for vertical stress under such loading, see Tabulated Values, chapter on Stress, page 20.

$$\Sigma Ff=V_x=w(\tfrac{1}{2}l-x).$$

x being measured from an origin at either support; the weakest section will occur where V_x is greatest, that is when $x=0$; or at either support; so that $V=\frac{1}{2}wl$.

Also the safe shearing strain is RS (see p. 88), where $R=$ safe resistance to shearing of the material.

$S=$ sectional area of the girder, or sectional area of its web.

So that the equation of safety at the ultimate is

$$\tfrac{1}{2}wl=RS; \text{ or } w=\frac{2RS}{l}.$$

In solid continuous sections of beams, girders, or tubes, the vertical stresses being far less than the horizontal

stresses, such beams, &c., as are safe as regards the latter have excess of strength to resist the former.

In girders of hollow rectangle or of **I** section, the web must have sufficient section, according to the above equation.

Braced girders follow an analogous law; the detail, which may be complicated, will be referred to under that special type.

3. Conditions of stiffness:

As explained in Chapter III., the horizontal forces alone cause deflexion; and, as before, their general value at any section, distant x from one support, is $\frac{1}{2}wx(l-x)$.

Using the same symbols as before, also

E=modulus of elasticity of material;

ρ=radius of curvature during deflexion;

x and y, the co-ordinates of any point p in the neutral axis of the deflected beam measured from the origin at A.

and applying the principle explained in Chapter III., page 102, we have

$$\frac{1}{2}wx(l-x)=\frac{EI}{\rho}=-EI\delta^2_x y;$$

or
$$EI.\delta^2_x y=\frac{1}{2}wx(x-l);$$

or
$$EI.y={}_0^x\!\int_x \, {}_{\frac{1}{2}l}^{x}\!\int_x \tfrac{1}{2}wx(x-l).$$

In the first integration, between limits x and $\frac{1}{2}l$, it will be observed that at the latter limit $\delta_x y=0$, as y then has a maximum,

$$\therefore EIy={}_0^x\!\int_x \tfrac{1}{2}w.\left\{\tfrac{1}{3}(x^3-\tfrac{1}{8}l^3)-\tfrac{1}{2}l(x^2-\tfrac{1}{4}l^2)\right\}.$$

In the next integration between x and 0, it will be observed that when $x=0$, $y=0$;

$$\therefore E\,I\,y=\tfrac{1}{2}w.\left\{\tfrac{1}{3}(\tfrac{1}{4}x^4-\tfrac{1}{8}l^3x-\tfrac{1}{2}l(\tfrac{1}{3}x^3-\tfrac{1}{4}l^2x)\right\};$$

or
$$E\,I\,y=\tfrac{1}{24}wx(x^3-2lx^2+l^3)\,;$$

the equation to the curve of the neutral axis giving the deflection y anywhere.

The flexure ξ is the greatest value of y, which occurs at midspan; that is, when $x=\frac{1}{2}l$, hence

$$\xi=\frac{5\,.\,wl^4}{384EI}.$$

With a solid rectangle, $\xi=\dfrac{5wl^4}{32bd^3.E}$.

With a solid ellipse, $\xi=\dfrac{5wl^4}{6\pi bd^3E}$.

With a hollow rectangle, $\xi=\dfrac{5wl^4}{32(bd^3-b_1d_1^3)E}$.

With a hollow ellipse, $\xi=\dfrac{5wl^4}{6\pi(bd^3-b_1d_1^3)E}$;

and so on with other sections, as already explained.

Solution Number 2.—Free girder of isotropic material of uniform and of symmetrical section under a collected load.

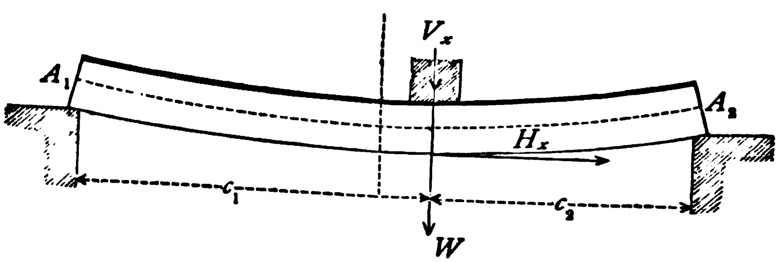


FIGURE 2.

1. The condition of sufficient strength to safely resist horizontal stress.

Let H_x be the horizontal stress at any section, whose abscissa is x;
R the safe resistance of the material;
I the moment of inertia of the cross-section;
a the axial distance of the extreme lamina;
l the span between points of support;
W the collected load.

If the load be collected at a point p, distant c_1 and c_2 respectively from the supports A_1 and A_2, the weakest section will necessarily be at p. For this point, according to Chapter on Stresses Table at page 20,

$H_x = \frac{W.c_1.c_2}{l}$; and the strain that may be safely induced at the ultimate $= \frac{RI}{a}$,

hence the equation of safety is $W = \frac{l.RI}{ac_1c_2}$.

And if W be placed at midspan, $c_1 = c_2 = \frac{1}{2}l$; then $W = \frac{4RI}{al}$.

In this case, with a solid rectangular section $W = \frac{2bd^2R}{3l}$

with a solid elliptic section, $W = \frac{\pi bd^2R}{8l}$;

with a hollow rectangular section, $W = \frac{2(bd^3 - b_1d_1^3)R}{3ld}$;

with a hollow elliptic section, $W = \frac{\pi(bd^3 - b_1d_1^3)R}{8ld}$.

And the remarks about sections in the last solution Number 1 apply here.

2. Conditions of sufficient strength to safely resist vertical stress.

The value of V_x at the point p, is, by table, page 20, $V_x = W.\frac{c_1}{l}$.

The safe shearing strain is, by Chapter III., equal to RS, where

R is the safe resistance to shearing of the material;

S is the web section.

So that the equation of safety is

$$W=\frac{c_1 RS}{l}.$$

The remarks on shearing strain in the last solution Number 1 apply here generally.

3. Conditions of stiffness.

A concentrated load will evidently produce most deflexion when placed at midspan. Let us, under this condition, treat each half-span as loaded with half the load. Also let us take the converse of the actual case; that is, let us take the midspan to be after deflexion a fixed point, but unloaded, while a tensile stress $\frac{1}{2}W$ is supposed to act vertically upwards at A; the condition regarding deflexion will remain as before.

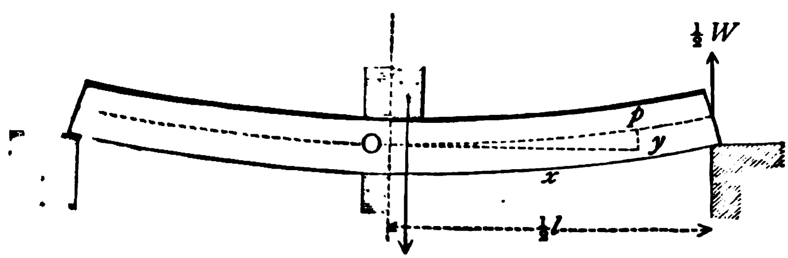


FIGURE 3.

Taking O midspan (deflected) as the origin of co-ordinates x and y; to determine the deflexion of any point p on the neutral axis.

First, as by the tabulated stresses H_x is $\frac{1}{2}Wx$, now that

the origin is at midspan, H_x becomes $\frac{1}{2}W(\frac{1}{2}l-x)$; also putting E=modulus of elasticity of material, ρ=radius of curvature from deflexion, and applying the principles of deflexion mentioned in Chapter III., page 104, we have

$$\tfrac{1}{2}W(\tfrac{1}{2}l-x)=\frac{EI}{\rho}=\partial_x^2 y.EI;$$

$$\therefore\ EI.y={}_0^x\!\int_x {}_0^x\!\int_x \tfrac{1}{2}W(\tfrac{1}{2}l-x).$$

In the first integration, it will be observed that when x=0, y=0, and hence also $\partial_x y$=0.

In the next integration, when x=0, y=0;

$$\therefore EIy={}_0^x\!\int_x \tfrac{1}{2}W(\tfrac{1}{2}lx-\tfrac{1}{2}x^2)$$

$$=\tfrac{1}{4}W(\tfrac{1}{2}lx^2-\tfrac{1}{3}x^3)=\tfrac{1}{24}Wx^2(3l-2x);$$

the equation to the curve of the neutral axis, whence the deflexion may be found for any abscissa.

The flexure ξ, or greatest deflexion, is the value of y, which occurs when $x=\frac{1}{2}l$,

hence $$\xi=\frac{Wl^3}{48\,EI}.$$

With a solid rectangle, $\xi=\dfrac{Wl^3}{4\,bd^3.E}$;

with a solid ellipse, $\xi=\dfrac{4\,Wl^3}{3\pi bh^3E}$;

with a hollow rectangle, $\xi=\dfrac{Wl^3}{4(bd^3-b_1d_1^3)E}$;

with a hollow ellipse, $\xi=\dfrac{4Wl^3}{3\pi(bh^3-b_1h_1^3)E}$.

The remarks about sections, **I** section, and unsymmetrical section, in Solution Number 1, apply here also.

Solution Number 3.—Free girder of isotropic material of uniform and of symmetrical section under twofold loading, that is, both equally distributed loading and a collected load.

1. Safe strength under horizontal stress.

From considering the mode applied in solutions (1) and (2) with these loads separately, it is clear that we can equate

$$H_x + H'_x = \frac{RI}{a},$$

where H_x is the horizontal stress due to one sort of load, and H'_x that due to the other.

Besides, we may alternatively adopt the following mode:

Let $W_1 = wl$, the total distributed load,

W = the collected load at midspan;

and by comparing the results of Solutions Numbers 1 and 2, we observe that W produces half the effect of W_1 in strain; hence with twofold loading we have, employing otherwise corresponding terms,

$$W + \tfrac{1}{2}W_1 = \frac{4RI}{al}$$

as the equation of safety.

2. Safe strength under vertical stress.

Considering the modes employed in Solutions Numbers 1 and 2; the two sets of stresses must be summed, and adopting terms similar to those before used, we have

$$V_x + V'_x = RS.$$

For values of V_x employ Table at p. 20, Stresses.

3. Conditions of stiffness.

For these we might employ the sum of the two sets of horizontal stresses, H_x and H'_x, and reintegrate. But

from observing the results of Solutions 1 and 2, and using the symbols W_1 and W respectively again, the effect of W' as regards flexure is that of $\frac{5}{8}W_1$; hence with twofold loading we have $\xi=\frac{(W+\frac{5}{8}W_1)l^3}{48EI}$.

If deflexions at other points be required, the complete working out is necessary.

The remarks on various sections in Solution 1 apply here also in a general sense.

Solution Number 4.—Free girder or beam of uniform but of unsymmetrical section; loaded in various ways.

Assuming the results of the last three solutions to hold with symmetrical sections of isotropic material, as timber and wrought iron approximately are, it is now needful to see how far any modifications of them can apply under other conditions of section, with other material, as cast iron.

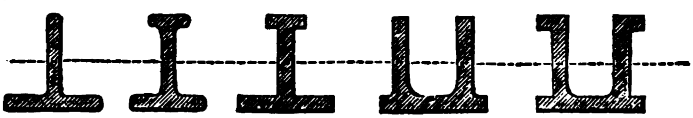


FIGURE 4.

It is evident that although the compressive and tensile resistances of cast iron are unequal, yet by a suitable balance of the upper and lower parts of a cast-iron section it may be so designed as to be an isotropic section, that is to say, presenting equal resistance above and below its neutral axis.

The forms used with such object are the following. They are necessarily so placed that the greater flange or member will be strained in tension, and their proportions depend on the principle of isotropy.

The Hodgkinson section is one of these, a I section,

with the greater flange area six times the smaller flange area; other ratios are used, as 4 to 1, &c., in accordance with the material to be actually used, and the inverse ratios of its tensile and compressive resistances; for this purpose the neutral axis is supposed to pass through the centre of gravity of the whole section.

Granting the isotropy of a section, and this position of the neutral axis as absolutely correct, we can then express the moment of resistance of the section, or induced strain, in the form

$\frac{R_c I_c}{a_c} + \frac{R_t I_t}{a_t}$; the subscript denoting the nature of the strain; or, if greater coarseness is permitted, in the form

either $\frac{R_c}{a_c}.I$ or $\frac{R_t}{a_t}I.$, using the less of the two; or,

either $R_c S_c.d_1$ or $R_t S_t d_1$, where d_1 is the effective depth of girder, and S the section strained; or finally $\frac{RI}{a}$ for the whole section; where R is the least favourable of the two safe resistances R_c and R_t; I is the moment of inertia of the whole section, and a is the axial distance of the furthest lamina, whether in tension or compression.

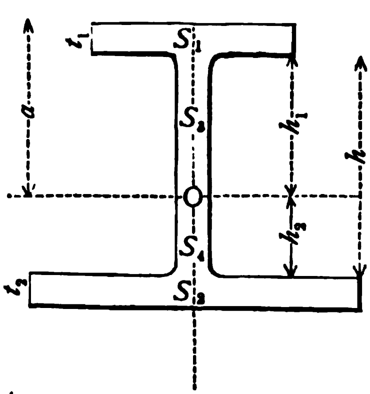


FIGURE 5.

For instance, in the section in the figure, as we have

$a_t : a_c : a_t + a_c :: R_t : R_c : R_t + R_c$

we may obtain

$$I = \left\{(S_1 + \tfrac{1}{3}S_3)h_1^2 + (S_2 + \tfrac{1}{4}S_4)\right\}h_2^2;$$

$$a = \frac{R_t(2S_2 + S_4)(h + t_1) + R_c(S_1 + S_3)t_1 + R_t S_3 t_2}{R_t(2S_2 + S_4) + R_c(2S_1 + S_3)};$$

Also if the isotropy is assured in the section we may put these in the forms

$$I=\tfrac{1}{12}\left\{S_1t_1^2+S_2t_2^2+(S_3+S_4)h^2\right\}+\frac{h^2}{4S}\left\{4S_1S_2+(S_1+S_2)(S_3+S_4)\right\};$$

$$a=\frac{1}{S}\left\{(t_2+\tfrac{1}{2}h)(S_3+S_4)+(\tfrac{1}{2}t_1+t_2+h)S_1+\tfrac{1}{2}t_2S_2\right\};$$

The same method may be applied to trough-sections, and these lengthy expressions are frequently shortened by neglecting the web in taking a value of I.

There is, however, no certainty in the belief that an unsymmetrical section, although by design isotropic, will have an invariable neutral axis passing through its centre of gravity; in the set up girder it may be near it only, hence it is necessary to allow for slight displacement of the neutral axis, and to estimate the sectional resistance under the unfavourable conditions resulting from it.

With such limitations the results of Solutions 1, 2, and 3 will apply to isotropic unsymmetrical sections.

Unsymmetrical sections, not designed to fulfil the condition of isotropy, cannot be theoretically dealt with in any way when they occur in girders of uniform section.

The use of such material in horizontal girders is generally restricted to single spans of small length.

Solution Number 5.—*Free girder or beam of uniform strength, under equally distributed load; the section symmetrical about the neutral axis, and of isotropic material.*

1. The conditions of sufficient strength to safely resist horizontal stress.

At midspan, where H_x is greatest, the condition will evidently be that at midspan of a girder of uniform section similarly loaded; in this case $w=\dfrac{8RI}{al^2}$; see Solution Number 1.

But with uniform strength throughout, the equation $B_x = \frac{1}{2}wx(l-x) = \frac{RI}{a}$ must hold at every section of the girder: and as B_x is variable, $\frac{I}{a}$ must vary.

With a section of symmetrical form, if the breadth vary, and the depth remain constant, then as $a=\frac{1}{2}d$, I will alone vary: but if the breadth remain constant and the depth varies, as is more commonly adopted, both a and I will vary. The curve either of breadths or of depths in either scale may be deduced and plotted; thus giving sufficient strength everywhere.

Taking the special case of a hollow rectangular section, as this also includes the **I** section, with constant breadth.

Let x and y be the co-ordinates from one point of support, A, of any point p in the curve of depths, then $a=\frac{1}{2}y$, and $I=\frac{1}{12}(by^3-b_1y_1^3)$

or neglecting the web-breadth, $I=\frac{1}{12}b(y^3-y_1^3)$;

inserting these above, we have $\frac{1}{2}wx(l-x)=\frac{Rb}{6y}(y^3-y_1^3)$,

or $y^2-\frac{y_1^3}{y}=\frac{3w}{Rb}(lx-x^2)$, the required equation to the curve, which is semi-elliptic.

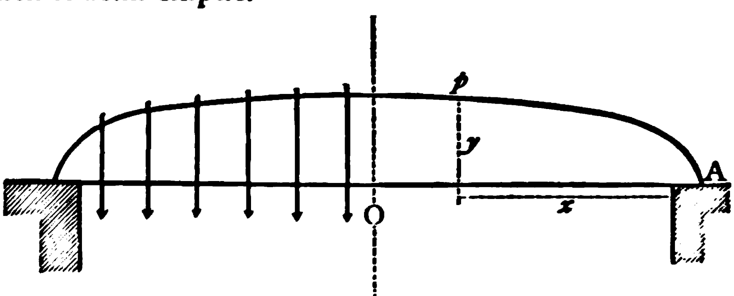


FIGURE 6.

This method will apply approximately if the dimensions of the circumscribing or representative rectangle of any symmetrical section be used.

2nd. Conditions of sufficient strength to safely resist vertical stress.

These do not produce results varying much from those practically necessary with girders of uniform section.

The equation is $V_x=w(\frac{1}{2}l-x)=RS$,
where R is the safe resistance of the material to shearing, and S is the web-section.

3rd. Conditions of stiffness.

The deflexion of a girder of uniform strength differs from that of a girder of uniform section.

First, let the depth be constant; the extreme axial distance a will be so also. The resistance offered at every section is invariable, hence curvature of the elastic curve is constant, and is a circular arc whose radius is

$$\rho=\frac{EI}{M}.$$

The constant moment of resistance M is the greatest value of M occurring in a similar girder having a uniform section; hence the corresponding value of I for the section at midspan also applies; and the above equation may be represented thus

$$\rho=\frac{EI_m}{M_m}\text{; where } M_m=\tfrac{1}{2}w\frac{l^2}{2}-\tfrac{1}{2}w\frac{l^2}{4}=\tfrac{1}{8}wl^2.$$

Hence, taking a fresh origin at O, we have

$$EI_m.y={}_0^x\int_x {}_0^x\int_x \tfrac{1}{8}wl^2={}_0^x\int_x \tfrac{1}{8}wl^2x=\tfrac{1}{16}wl^2x^2,$$

the general equation of curvature.

Also, as y is greatest when $x=\frac{1}{2}l$, the flexure $\xi=\frac{wl^4}{64EI_m}$.

Secondly, let the breadth be constant; then the depth and the extreme axial distance a will both be variable.

But having a constant value of $\frac{M.d}{I}$ everywhere throughout the girder, this is equal to $\frac{M_m d_m}{I_m}$. Also the square of the depth varies with the moment of resistance.

$$\text{Hence } EI_m . \partial_x^2 y = M_m \left\{\frac{M_m}{M}\right\}^{\frac{1}{2}} = \tfrac{1}{8} wl^2 \left\{\frac{\frac{1}{8} wl^2}{\frac{1}{2} w(\frac{1}{4} l^2 - x^2)}\right\}^{\frac{1}{2}}$$

$$= \tfrac{1}{16} wl^3 . \frac{1}{(\frac{1}{4} l^2 - x^2)^{\frac{1}{2}}};$$

$$\therefore EI_m . \partial_x y = \tfrac{1}{16} wl^3 . \sin^{-1} \frac{x}{\frac{1}{2} l};$$

$$= \tfrac{1}{16} wl^3 \left\{\sin^{-1}\left(\frac{2x}{l}\right) \times 1 + \frac{x}{(\frac{1}{4} l^2 - x^2)^{\frac{1}{2}}} - \frac{2x}{2(\frac{1}{4} l^2 - x^2)^{\frac{1}{2}}}\right\}$$

$$\therefore EI_m y = \tfrac{1}{16} wl^3 \left\{x . \sin^{-1}\left(\frac{2x}{l}\right) + \tfrac{1}{2}(\tfrac{1}{4} l^2 - x^2)^{\frac{1}{2}} + C\right\};$$

when $x=0$, $y=0$, hence $C=-\frac{1}{2} l$;

$$\text{and } EI_m y = \tfrac{1}{16} wl^3 \left\{x \sin^{-1}\left(\frac{2x}{l}\right) + \tfrac{1}{2}(\tfrac{1}{4} l^2 - x^2)^{\frac{1}{2}} - \tfrac{1}{2} l\right\}$$

is the required equation to the curve.

Also, the flexure is the value of y when $x=\frac{1}{2} l$; hence $\xi = \frac{1}{64} wl^4 . \frac{(\pi - 2)}{EI_m}$.

Solution Number 6.—*Free girder or beam of uniform strength, under a collected load; the section symmetrical about the neutral axis, and of isotropic material.*

1. The conditions of safe strength under horizontal stress.

At the point of application of load where H_x is greatest, the condition will evidently be that at the similar point of a girder of uniform section similarly loaded; in this case

$W=\frac{l.RI}{a.c_1c_2}$; see Solution No. 2 for this value and for symbols.

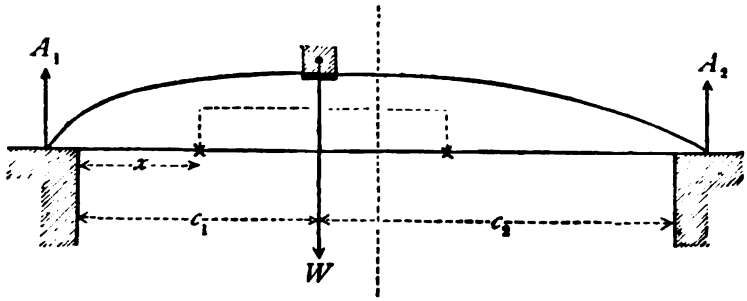


FIGURE 7.

But with uniform strength throughout, the equation $H_x = Wx\frac{c_2}{l} = \frac{RI}{a}$ must hold at every section of the girder on the left side of the load, near A_1; and $H_x = Wc_1\frac{l-x}{l} = \frac{RI}{a}$ on the right side of it near A_2, the origin in either case being at A_1. By shifting the origin, these two equations may be made to correspond, hence there is merely need to deal with one of them, calling c the distance of the load and x the variable abscissa of the section, from either origin chosen.

$$H_x = Wx\frac{c}{l} = \frac{RI}{a}.$$

With a section of symmetrical form, if the breadth vary, the depth may remain constant, then $a=\frac{1}{2}d$, and I will vary; but if the breadth remain constant and the depth varies, as is more commonly adopted, both a and I will vary. The curve either of breadths or of depths in either case may be deduced from the equation and plotted, thus obtaining sufficient strength everywhere.

Taking the special case of a hollow rectangular section,

as this also includes the **I** section, and adopting constant breadth, let x and y be the co-ordinates from A of any point in the curve of depths, then $a=\frac{1}{2}y$; also by neglecting the web-breadth, and putting $b=b_1$ in $I=\frac{1}{12}(bd^3-b_1d_1^3)$; this becomes $I=\frac{1}{12}b(y^3-y_1^3)$;

then $y^2-\dfrac{y_1^3}{y}=\dfrac{6Wcx}{blR}$; is the equation to the curve of depths, which is parabolic from either support up to the load. From this d_1 and $\frac{1}{2}(d-d_1)$ the depth of either flange may be obtained.

The collected load when placed at midspan will exert most stress; in that case $W=\dfrac{4RI}{al}$ is the condition at midspan, and $y^2-\dfrac{y_1^3}{y}=\dfrac{3Wx}{2bR}$ is the equation to the curve of depths for that special case.

2. Conditions of sufficient strength to safely resist vertical stress.

The two equations are

$$V_x=+W\frac{c_2}{l}=RS \text{ on the left of the load};$$

$$V_x=-W\frac{c_1}{l}=RS \text{ on the right of the load};$$

R being the safe resistance of material to shearing,

S the web section,

the results do not actually vary much from those for a girder of uniform section.

3. Conditions of stiffness.

The deflexion being evidently greatest when the load is at midspan, this case will be adopted.

First, let the depth be constant, then a, the extreme axial distance, is also constant. The moment of resistance being constant throughout, the curvature of deflexion will also

be constant, and is a circular arc, whose radius is $\rho=\frac{EI}{M}$. The constant moment of resistance is the greatest moment on a corresponding girder of uniform section similarly loaded, or $M_m=\frac{1}{4}Wl$; $\therefore \rho=\frac{EI_m}{M_m}$; in definite terms; I_m being the moment of inertia of the section at midspan.

Taking now an origin O at midspan,

$$EI_m.\partial_x^2 y=\tfrac{1}{4}Wl; \therefore EI_m.\partial_x y=\tfrac{1}{4}Wlx;$$

and $EI_m y=\frac{1}{8}Wlx^2$; the equation of curvature to the neutral axis.

Also ξ, the flexure, is the greatest value of y, which occurs when $x=\frac{1}{2}l$,

$$\therefore \xi=\frac{Wl^3}{32EI_m}.$$

Secondly, when the breadth is constant, the depth and the axial distance are both variable.

Then $\frac{Md}{I}$ is constant throughout, being $\frac{M_m d_m}{I_m}$; comprising the maximum midspan values for a corresponding girder of uniform section similarly loaded; also the square of the depth varies as the moment.

But $M_m=\frac{1}{4}Wl$; and $M=\frac{1}{2}W(\frac{1}{2}l-x)$ from the new origin at O, midspan, and is a variable;

$$\therefore EI_m\partial_x^2 y=M_m\left(\frac{M_m}{M}\right)^{\frac{1}{2}}=\tfrac{1}{4}Wl\left(\frac{\frac{1}{2}l}{\frac{1}{2}l-x}\right)^{\frac{1}{2}};$$

integrating twice between the limits x and o,

$$EI_m.\partial_x y=\tfrac{1}{2}Wl\{-(\tfrac{1}{2}l)^{\frac{1}{2}}(\tfrac{1}{2}l-x)^{\frac{1}{2}}+\tfrac{1}{2}l\};$$

$$EI_m y=\tfrac{1}{2}Wl\{+(\tfrac{1}{2}l)^{\frac{1}{2}}.\tfrac{2}{3}(\tfrac{1}{2}l-x)^{\frac{3}{2}}+\tfrac{1}{2}lx-\tfrac{2}{3}(\tfrac{1}{2}l)^2\};$$

$$=\tfrac{1}{8}Wl\{(\tfrac{1}{2}l)^{\frac{1}{2}}(\tfrac{1}{2}l-x)^{\frac{3}{2}}+\tfrac{3}{4}lx-(\tfrac{1}{2}l)^2\},$$

the equation to the curve of the neutral axis.

The flexure ξ, or greatest value of y, when $x=\frac{1}{2}l$, is $\xi=\frac{Wl^3}{24EI_m}$.

It will be noticed that some of the equations, obtained in relation to girders of uniform strength, involve two unknown quantities, and hence require repeated approximation. The inherent weight, being affected by variable dimensions, may also complicate these further.

Solution Number 7.—Free girder of uniform strength under twofold loading; that is, both equally distributed and a collected load; the section symmetrical about the neutral axis and of isotropic material.

1. Safe strength under horizontal stress.

From considering the equations and mode applied in Solutions 5 and 6 with the loads separately, it is evident that we may similarly apply the sum of the two sets of horizontal stresses,

$$H_x+H'_x=\tfrac{1}{2}wx(l-x)+Wx\frac{c}{l}=\frac{RI}{a},$$

and proceed as before to obtain either the variable depths or the variable breadths, as may be required. The greatest value of $H_x+H'_x$ will occur at some section whose position is dependent partly on that of the concentrated load; this would be shown in a diagram of the plotted values.

3. Safe strength under vertical stress.

The summation of the two sets of stresses gives

$$V_x+V'_x=w(\tfrac{1}{2}l-x)+W.\frac{c_2}{l}=RS \text{ to left of the load;}$$

$$\text{or}\quad V_x+V'_x=w(\tfrac{1}{2}l-x)-W\frac{c_1}{l}=RS \text{ to right of the load.}$$

R being the safe resistance of material under shear,
S the sectional area of the web.

The results in web area vary little actually from those for girders of uniform section similarly loaded. See also Solution Number 3.

3. Conditions of stiffness.

Having found the greatest value of $H_x + H'_x$, and determined a value of I_m, the moment of inertia of the section necessary for safe resistance at that point, the equation to the elastic curve and the flexure may be obtained in the same way as in Solutions 5 and 6.

Solution Number 8.—Free girder of uniform strength, but of unsymmetrical section loaded in various ways.

The unsymmetrical section when rendered isotropic by proportioning its upper and lower flanges in accordance with the compressive and tensile resistances of material that is not isotropic, has been treated in Solution No. 4.

It was there explained that in girders of uniform but unsymmetrical section, even when isotropic by design, the position of the neutral plane might practically vary to some small unknown degree. This uncertainty is still more troublesome in girders of uniform strength where the neutral axis is not horizontal; and the before-mentioned limitation, or allowance for variation of neutral axis, will also be more largely applied when dealing with these. The results will necessarily be merely approximative; the methods of arriving at them will be analogous to those employed in Solutions Nos. 5 and 6 for symmetrical sections of isotropic material.

I. With an equally distributed load, adopting the unsymmetrical section used in Solution 4 and the method

employed in Solution 5, we have generally at every section of the girder,

$\frac{1}{2}wx(l-x)=\frac{RI}{a}$; while at the greatest section $x=\frac{l}{2}$.

If the breadth of the section be constant and the depth varies; let x and y be the co-ordinates from either point of support of any point p in the curve or line of depths. To

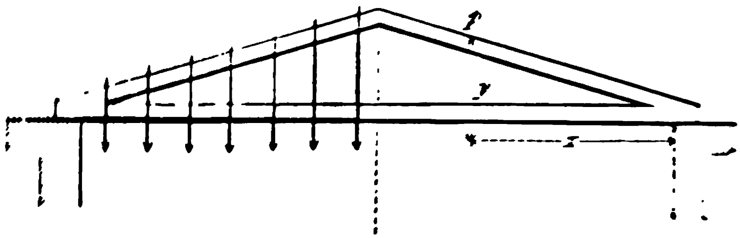


FIGURE 8.

reduce the expression for the variable $\frac{I}{a}$; taking the values from Solution 4, let $h=y$; $S_2=nS_1$; $S_3+S_4=t_3h=my$: and rejecting the web section

$$\frac{I}{a}=\frac{(S_1t_1^2+S_2t_2^2+my^2)(S_1+S_2+my)+\{12S_1S_2+(3S_1+3S_2)my\}y^2}{12\{(2t_3+y)my+(t_1+2t_2+2y)S_1+t_2S_2\}}$$

$$-\frac{1}{12}S_1\left\{\frac{(n+1)(t_1^2+nt_2^2)+12ny^2}{t_1+2y+t_2(n+2)}\right\}$$

$$\therefore wx(l-x)=\frac{1}{6}RS_1\left\{\frac{(n+1)(t_1^2+nt_2^2)12ny^2}{t_1+2y+t_2(n+2)}\right\}$$

an equation connecting x and y.

If the depth of the section be constant, and the breadths of the flanges vary to afford uniformity of strength, we may obtain the curvature of breadth. Let x and y be the co-ordinates from a point of support, y the breadth of one flange, ny the breadth of the other, at a horizontal distance x.

Taking the value of $\frac{I}{a}$ from Solution 4, and reducing, then $S_1=yt_1$; $S_2=nS_1=nyt_2$; $S_3+S_4=t_3h$; $a=h(n+1)$;

and neglecting the web-section,

$$\frac{I}{a}=\frac{y(t_1^2+nt_2^2)t_1+y\dfrac{nht_1}{n+1}}{12h(n+1)};$$

$$=\tfrac{1}{12}y.\frac{n+1}{h}(t_1^2+nt_2^2)t_1+ynht_1;$$

$$\therefore\ x(l-x)=\frac{Ryt_1}{6wh}\left\{(n+1)(t_1^2+nt_2^2)+12nh^2\right\},$$

the equation to the curvature of the lower flange, a parabola having a parameter equal to the coefficient of y in the preceding equation.

At midspan half the breadth of this flange is

$$\frac{3hl^2w}{2Rt_1\{(n+1)(t_1^2+nt_2^2)+12nh^2\}}.$$

For the curvature of the other flange, substitute $\frac{yt_2}{nt_1}$ for y in the former equation; also for its half-breadth at midspan, multiply the above given half-breadth of the lower flange by the same quantity.

The deflexion in these two cases will be that given generally in Solution Number 5.

With constant breadth

$$EI_m y=\tfrac{1}{16}wl^3\left\{x\sin^{-1}\frac{2x}{l}+\tfrac{1}{2}(\tfrac{1}{4}l^2-x^2)^{\frac{1}{2}}-\tfrac{1}{2}l\right\};\ \xi=\frac{(\pi-2)wl^4}{64.EI_m}.$$

With constant depth

$$EI_m y=\tfrac{1}{16}wl^2x^2;\ \text{and}\ \xi=\tfrac{1}{64}\frac{wl^4}{EI_m}.$$

II. With a collected load.

Adopting the same unsymmetrical section used in Solution 4, and the method and terms used in Solution 6,

we have generally with an origin at either support, as the condition at any section,

[illegible] generally and [illegible] at the greatest section, which occurs at the load.

If the breadth of the section be constant, we may obtain the curve of variation in depth.

using [illegible]

[illegible]

an equation connecting x and y, that affords the possible values of y the depth for a range of values of x taken from either support up to the load.

The greatest depth will correspond to a value of $x=l-c$ or $=c_1$, when the above $c=c_2$; so that that depth is [illegible]

If the depth of the section be constant, we may obtain the curve of variation in breadth, putting y for the variable breadth of one flange, ny that of the other at any horizontal distance x from either support,

using
$$\frac{I}{a}=\frac{yt_2}{12h}\left\{(n+1)(t_1+nt_2^2)+12nk^2\right\};$$

$$\frac{Wc}{l}.x=\frac{Ryt_2}{12h}\left\{(n+1)(t_1+nt_2^2)+12nk^2\right\};$$

an equation connecting x and y, giving a straight line, and yielding values of y up to the load. For the greatest breadth of this flange put $x=l-c$ or $=c_1$, when the above $c=c_2$, and solve the above equation for y.

Also to obtain the greatest breadth of the other flange put $nb_1d_1=b_2d_2$, where b_1 and b_2 are the greatest corresponding

breadths, and d_1 and d_2 are the corresponding depths of the two flanges.

The deflexions in these two cases will be those already given in Solution No. 6.

With constant breadth

$$EI_m y=\tfrac{1}{2}Wl\left\{(\tfrac{1}{2}l)^{\frac{1}{2}}.\tfrac{2}{3}(\tfrac{1}{2}l-x)^{\frac{1}{2}}-\tfrac{1}{2}lx-\tfrac{2}{3}(\tfrac{1}{2}l)^2\right\};\ \xi=\tfrac{1}{24}\frac{Wl^3}{EI_m}.$$

With constant depth $EI_m y=\frac{1}{8}Wlx^2$; and $\xi=\dfrac{Wl^3}{32EI_m}$.

These unsymmetrical sections, used in cast-iron, are merely employed in single spans, and seldom in a large one. Under such conditions a direct modulus of transverse strength and a direct modulus of flexure would be the preferable mode of treatment, with allowance for sectional variety; but experimental investigation is still deficient on these points, and quality in castings varies much.

CANTILEVERS.

Solution Number 9.—Cantilever of uniform and symmetrical section of isotropic material, having an equally distributed load.

1. The condition of safety under horizontal stress.

Let ΣFf be the sum of the moments of external force with reference to any section of the cantilever.

R be the safe resistance of the material, either to compression or tension.

I the moment of inertia of any cross-section.

a the extreme axial distance in the section, or distance from the neutral axis to the furthest lamina.

l the length of the cantilever.

w the load per unit of length of l.

Taking an origin O at the point of support and of fixture, x and y the rectangular co-ordinates.

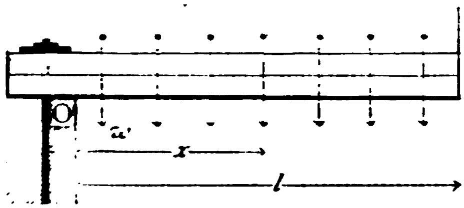


FIGURE 9.

In order that there may be sufficient strength throughout the cantilever its weakest section must be safe, and at that section the equation of stress and safe strain at the ultimate will be expressed by $\Sigma Ff = H = M$, the moment of resistance.

From Chapter II. on Stresses, and the tabulated values at page 22 of it, we have for the general expression for horizontal stress on a cantilever uniformly loaded

$$H_x = \tfrac{1}{2} w (l - x)^2,$$

when x is measured from the fixed end.

We have also from the Chapter III. on Strains, page 97, &c., $M = \dfrac{RI}{a}$,

hence the equation of safety is

$$\tfrac{1}{2} w (l - x)^2 = \frac{RI}{a},$$

and the weakest section will evidently be at a point where H is greatest or M is least. But as the terms involved in M are constants the weakest section occurs where H_x is greatest, which is when $x = 0$, or at the fixed end. In that case the equation of safety becomes

$$\tfrac{1}{2} w l^2 = \frac{RI}{a}; \text{ or } w = \frac{2RI}{al^2}.$$

With all symmetrical sections of depth d and of breadth (b), we have $a = \tfrac{1}{2} d$.

With a solid rectangle, $I=\frac{1}{12}bd^3$; $w=\frac{bd^2R}{3l^2}$.

With a solid ellipse, $I=\frac{1}{64}\pi ba^3$; $w=\frac{\pi ba^2.R}{16l^2}$.

The square and circle are merely particular forms of these when $b=d$.

With the hollow rectangle, $I=\frac{1}{12}(bd^3-b_1d_1^3)$; where b_1 and d_1 are the breadth and depth of the hollow, and $w=\frac{R(bd^3-b_1d_1^3)}{3dl^2}$.

With the hollow ellipse, $I=\frac{1}{64}\pi(bd^3-b_1d_1^3)$; and $w=\frac{\pi R(bd^3-b_1d_1^3)}{16dl^2}$.

The **I** section may be treated as a hollow rectangle having the web-depth $=d_1$, and the sum of the breadths of the two side-voids $=b_1$.

With other symmetrical sections the values of I and a must be applied in the general formula.

2. Conditions of sufficient strength to safely resist vertical stress.

The general expression for vertical stress under uniform load (see tabulated values, page 22 of Chapter II., on Stress) $=wx$, when the origin is at the free end, or $w(l-x)$ when the origin is at O, the fixed end; in this case

$$\Sigma Ff=V_x=w(l-x)$$

the weakest section will be where V_x is greatest, that is where $x=0$, or at the fixed point.

The safe shearing strain on the section $=RS$. See Chapter III. on Strain; where R is the safe shearing strain on the material, and S is the sectional area strained, in this case that of the web, or the vertical section.

The equation of safety is therefore

$$wl=RS\text{; or } w=\frac{RS}{l}.$$

In solid continuous sections of cantilevers, the vertical stresses are far less than the horizontal stresses, so that beams that are safe as regards the latter are also safe regarding the former.

In box or **I** sections the webs require the above formula.

Braced cantilevers follow an analogous law: the detail, which may be complicated, will be referred to under girders, and cantilevers of that type.

3. Conditions of stiffness.

As explained in Chapter III., the horizontal forces alone cause deflexion, hence, as before, $H_x=\frac{1}{2}w(l-x)^2$ also E = modulus of elasticity of the material,

ρ = radius of curvature from deflexion,

and as $\frac{I}{\rho}=-\partial_x^2 y$; x and y being taken from the same origin O, as before, we have

$$EI(-\partial_x^2 y)=\frac{1}{2}w(l-x); \text{ or } EI.y={}_0^x\int_x {}_0^x\int_x -\frac{1}{2}w(l-x).$$

Integrating twice between the limits x and o, and noticing that when x=o, $\partial_x y$=o, and y=o;

$$EIy={}_0^x\int_x \frac{1}{2}w\left\{-\frac{1}{3}(l-x)^3+\frac{1}{3}l^3\right\}$$
$$=\frac{1}{2}w\left\{\frac{1}{12}(l-x)^4+\frac{1}{3}l^3x-\frac{1}{12}l^4\right\}.$$

the equation to the curve of the neutral surface.

The flexure ξ, or greatest value of y, occurs when $x=l$;

$$\therefore \xi=\frac{wl^4}{8EI};$$

in which the value of I for any section may be substituted.

Solution Number 10.—*Cantilever of uniform and of symmetrical section, of isotropic material, under a collected load.*

As a collected load will evidently strain the cantilever most when placed at the free end, this case will be adopted.

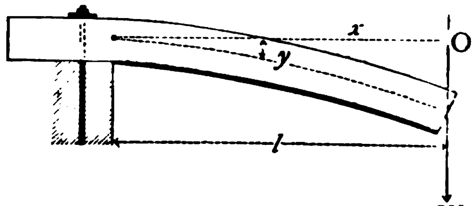


FIGURE 10.

1. Sufficient strength to safely resist horizontal stress.

Taking O the origin at the free extremity,

let H_x be the horizontal stress at any section whose abscissa is x,

R the safe resistance of the material either to compression or tension,

I the moment of inertia at any cross-section,

a the extreme axial distance in the section,

l the length of the cantilever,

W the concentrated load,

Then the equation of stress and strain is $H=M$; and the extreme value of $H_x=-W_x$ is Wl, the weakest section being at the fixed end. And by Chapter III., on Strains, page 97, $M=\dfrac{RI}{a}$,

hence $$Wl=\frac{RI}{a}; \text{ or } W=\frac{RI}{al},$$

a condition corresponding to that of half a free girder resting on two supports.

The values of I and of a for any symmetrical section may be here substituted, as in preceding case.

2. Strength to safely resist vertical stress.

The extreme value of V_x, the vertical stress, is $-W$. See tabulated values, page 22 of Chapter II., on Stress.

The resistance is RS, where R is the safe resistance of material under shearing, S is the area of the vertical section or web.

The equation of safety is $W=RS$.

3. Condition of stiffness.

If E=modulus of elasticity of the material,

ρ=radius of curvature from deflexion.

Keeping the origin O at the free end,

$$H_x=-W_x\text{; and }\frac{1}{\rho}=-\delta^2_x y\text{;}$$

hence
$$EIy={}_l^x\!\int {}_l^x\!\int_x Wx={}_l^x\!\int_x \tfrac{1}{2}W(x^2-l^2)$$
$$=\tfrac{1}{2}W.\{\tfrac{1}{3}(x^3-l^3)-(l^2x-l^3)\},$$

the integration being twice made between the limits x and l, and observing that when $x=l$, $y=0$, and $\delta_x y=0$.

The above is the equation to the neutral axis; and the flexure ξ is the greatest value of y, which occurs when $x=0$,

hence
$$\xi=\frac{Wl^3}{3EI}.$$

The value of I for any symmetrical section can be substituted in these equations, as in the preceding case.

Solution Number 11.—*Cantilever of uniform and symmetrical section of isotropic material, under twofold loading equally distributed and a concentrated load.*

1st. Safe strength against horizontal stress.

$$-(Wx+\tfrac{1}{2}wx^2)=H_x+H'_x=\frac{RI}{a},$$

or putting $wl=W_1$; and noticing that $\frac{1}{2}W_1$ is a representative load under horizontal stress with regard to the weakest section, we obtain

$W+\frac{1}{2}W_1=\dfrac{2RI}{al}$ for the equation of safety.

2nd. Condition of safety under vertical stress.

$$V_x + V'_x = w(l-x) + W = RS,$$

where R is the safe resistance to shearing, S is the area of web section.

3rd. Condition of stiffness.

From Solutions 9 and 10 we see that if $wl = W_1$, the representative collected load for equivalent flexure would be $\frac{3}{8} W_1$, hence

$$\xi = \frac{(W + \frac{3}{8} W_1) . l}{3EI} .$$

For the curvature, y may be evaluated in the same way as before, but using $Wx + \frac{1}{2} wx^2$ as the total horizontal stress.

Solution Number 12.—*Cantilever of uniform but of unsymmetrical section, loaded in various ways.*

Let the unsymmetrical section be designed so as to be isotropic; see remarks in Solution Number 4, and under the limitations there expressed, let us employ the unsymmetrical section there used for further exemplification in this case.

1. Making use of the results of Solutions 9, 10, and 11, we have for conditions of strength to resist horizontal stress, under the three mentioned modes of loading,

$$\tfrac{1}{2} wl^2 = \frac{RI}{a} ; \; Wl = \frac{RI}{a} ; \; (W + \tfrac{1}{2} W_1) l = \frac{RI}{a} ;$$

where R is the least favourable of the two safe resistances to compression or tension of the material.

$$I = \{ (S_1 + \tfrac{1}{3} S_3) h_1^2 + (S_2 + \tfrac{1}{4} S_4) h_2^2 \} ;$$

$$a = \frac{1}{S} \{ t_2 + \tfrac{1}{2} h) (S_3 + S_4) + (\tfrac{1}{2} t_1 + t_2 + h) S_1 + \tfrac{1}{2} t_2 S_2 \} .$$

2. Similarly also in shearing strain for the three cases,

$$wl=RS\,;\ W=RS\,;\ wl+W=RS.$$

3. Conditions of stiffness.

The results of Solutions 9, 10, and 11 with the above values of I will afford the theoretical deflections.

Solution Number 13.—*Cantilever of uniform strength under equally distributed load.*

1st. Condition of safety under horizontal stress.

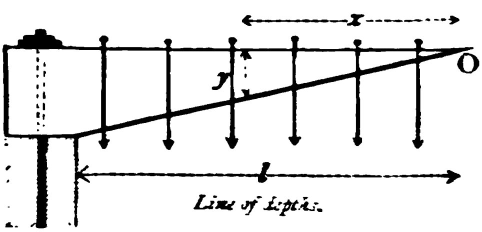


FIGURE 11.

At the fixed end $\frac{1}{2}wl^2=\frac{RI}{a}$.

Also, taking the origin of co-ordinates at O the free end, $\frac{1}{2}wx^2=\frac{RI}{a}$ is a condition that must hold everywhere. Taking a hollow rectangular or I section for example where $I=\frac{1}{12}b(d^3-d_1^3)$; putting $b_1=b$.

When the breadth is constant, and d varies

$$\frac{1}{2}wx^2=\frac{b(y^3-y_1^3)R}{6y}\ ;\ \text{or}\ y^2-\frac{y_1^3}{y}=\frac{3wx^2}{bR}\ :$$

the equation to the line of depths; or if preferred this variation may be confined to the depth of flange $\frac{1}{2}(d-d_1)$.

When the depth is constant.

$\frac{1}{2}wx^2=\frac{R}{6d}y.(d^3-d_1^3)$: is the equation to the line of breadths.

2nd. The condition of safety under vertical stress differs but slightly from that for cantilevers of uniform section. See Solution Number 9.

3rd. Conditions of stiffness.

When the breadth is constant; d and a are both variable; and the condition of uniform strength gives $\frac{Md}{I}$ constant at every section $=\frac{M_m d_m}{I_m}$, the extreme value at the weakest position. Also the square of the depth will vary with the variable moment of resistance.

Taking now the origin O at the fixed end of the cantilever, $M=\frac{1}{2}w\,(l-x)^2$; $M_m=\frac{1}{2}wl^2$;

and $-EI_m.\partial_x^2 y=M_m\left\{\frac{M_m}{M}\right\}^{\frac{1}{2}}=\frac{1}{2}wl^2\left(\frac{l}{l-x}\right)$.

Integrating twice between x and o,

$$-EI_m.\partial_x y=\tfrac{1}{2}\,wl^2.\log\left(1-\frac{x}{l}\right)$$

$$=\tfrac{1}{2}\,wl^2\left\{\log\left(1-\frac{x}{l}\right)+\frac{l}{l-x}-\frac{l}{l-x}\right\};$$

$$-EI_m y\quad=\tfrac{1}{2}\,wl^3\left\{(l-x)\,.\log\left(1-\frac{x}{l}\right)+x\right\}.$$

Also $\xi=y$, when $x=l$; $\therefore\ \xi=\frac{wl^4}{2EI_m}$.

When the depth is constant, then a is constant and the breadth varies. The moment of resistance M is constant, and the curvature will be also constant. M will be the greatest moment of resistance of a corresponding cantilever of uniform section, which is $M_m=\frac{1}{2}wl^2$. Using the fixed point for origin, $EI_m y=\int_0^x\int_0^x\frac{1}{2}wl^2=\int_0^x\frac{1}{2}wl^2x=\frac{1}{4}\,wl^2x^2$, the equation to the curve of the neutral axis.

Also $\xi=y$, when $x=l$; $\therefore\xi=\frac{wl^4}{4EI_m}$;

in which equations the value of I_m for the weakest position, or largest section, can be introduced.

Solution Number 14.—*Cantilever of uniform strength, under a collected load placed at the free end.*

1. Condition of safety under horizontal stress.

At the fixed end $Wl=\frac{RI}{a}$.

Also taking the origin of co-ordinates at O the free end, $Wx=\frac{RI}{a}$ is a condition that must hold everywhere. Taking a hollow rectangle or **I** section for example where $I=\frac{1}{12}b\,(d^3-d_1^3)$; and putting $b_1=b$,

When the breadth is constant and d varies,

$$Wx=\frac{b.(y^3-y_1^3)R}{6y};\ \text{ or } y^2-\frac{y_1^3}{y}=\frac{6Wx}{bR};$$

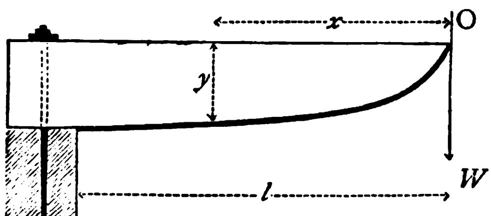


FIGURE 12.—Curve of depths.

the equation to the curve of depths, a parabola with the vertex at O; or if preferred this variation may be confined to the two flanges; each of which is equal to $\frac{1}{2}(d-d_1)$.

When the depth is constant,

$Wx=\frac{R}{6d}.y(d^3-d_1^3)$ is the equation to the line of breadths.

2. Condition of safety under vertical stress.

This practically differs little from that for a cantilever of uniform section. (See Solution Number 10.)

3. Condition of stiffness.

When the breadth is constant, d and a are both variable, and the condition of uniform strength gives $\frac{Md}{I}$ constant everywhere $=\frac{M_m d_m}{I_m}$, the extreme value at the weakest position; also the square of the depth varies with the moment of resistance.

Taking now the origin O at the fixed end of the cantilever, $M=W(l-x)$; $M_m=Wl$.

$$\therefore -EI_m.\partial_x^2 y = M_m\left\{\frac{M_m}{M}\right\}^{\frac{1}{2}} = Wl.\left(\frac{l}{l-x}\right)^{\frac{1}{2}};$$

integrating twice between limits x and o,

$$-EI_m.\partial_x y = Wl^{\frac{3}{2}}.\left\{-2(l-x)^{\frac{1}{2}}+2l^{\frac{1}{2}}\right\};$$

$$-EI_m y = Wl^{\frac{3}{2}}\left\{+\tfrac{4}{3}(l-x)^{\frac{3}{2}}+2l^{\frac{1}{2}}x-\tfrac{4}{3}l^{\frac{3}{2}}\right\}$$

$$=\tfrac{2}{3}Wl\left\{2l^{\frac{1}{2}}(l-x)^{\frac{3}{2}}+3lx-2l^2\right\}.$$

Also the flexure $\xi=y$, when $x=l$; $\therefore\ \xi=\frac{2Wl^3}{3EI_m}$.

In these equations the extreme value I_m for the greatest section must be substituted.

When the depth is constant, a is constant, and the breadth varies. The moment of resistance M is constant; the curvature is also constant; and M is the greatest moment occurring in a corresponding cantilever of uniform section, which is $M_m=Wl$. Using the fixed point for origin,

$$EI_m y={}_0^x\!\int {}_0^x\!\int_x Wl={}_0^x\!\int Wlx=\tfrac{1}{2}Wlx^2;$$

the equation to the curve of the neutral axis.

Also $\xi = y$ when $x = l$; $\therefore \xi = \frac{Wl^3}{2EI}$;

when the value of I_m for the greatest section can be introduced.

Solution Number 15.—*Cantilever of uniform strength under twofold loading.*

1. Condition of safety under horizontal stress.

At the fixed end $\frac{1}{2}wl^2 + Wl = \frac{RI}{a}$.

Also taking the origin of co-ordinates at O the free end $\frac{1}{2}wx^2 + Wx = \frac{RI}{a}$ is the general condition throughout.

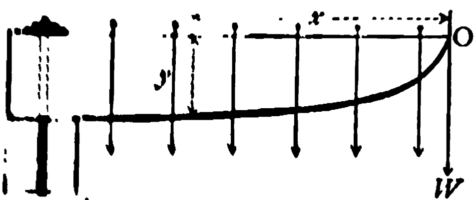


FIGURE 13. Curve of depths.

Let us adopt a hollow rectangle or **I** section, for example, where $I = \frac{1}{12}b(d^3 - d_1^3)$; when $b_1 = b$, that is, neglecting the web.

When the breadth is constant, and d varies,

$$\frac{1}{2}wx^2 + Wx = \frac{bR}{6y}\left(y^3 - y_1^3\right);\ \text{or } y^2 - \frac{y_1^3}{y} = \frac{3w}{bR}\left(\frac{2W}{w}x + x^2\right)$$

the equation to the curve of depths, a hyperbola with its vertex at O; or if preferred, this variation may be confined to the flanges, each of which is equal to $\frac{1}{2}(d - d_1)$.

When the depth is constant,

$\frac{1}{2}wx^2 + Wx = \frac{R}{6d}y(d^3 - d_1^3)$ is the equation to the line of breadths.

2. Condition of safety under vertical stress.

This practically differs little from that of a similar cantilever of uniform section. (See Solution No. 11.)

3. Condition of stiffness.

When the breadth is constant, the same equation, $EI_m.\partial_x^2 y = M_m\left(\frac{M_m}{M}\right)^{\frac{1}{2}}$ can be worked out as before, but

$$M_m = \tfrac{1}{2}wl^2 + Wl = \tfrac{1}{2}W_1 l + Wl = \tfrac{1}{2}l(W_1 + 2W);$$

and $M = \frac{1}{2}w(l-x)^2 + W(l-x)$.

If the flexure ξ alone be required, it may be obtained by observing that under these conditions the effect on ξ of a distributed load $W_1 = wl$ is $\frac{3}{4}$ of that of a collected load. See Solutions 12 and 13.

Hence with twofold loading, and constant breadth,

$$\xi = \frac{2(\frac{3}{4}W_1 + W)}{2EI_m} = \frac{3W_1 + 4W}{6EI_m}.$$

When the depth is constant, the corresponding equation is $EI_m\partial_x^2 y = M_m = \frac{1}{2}l(W_1 + 2W)$; whence we obtain

$$EI_m y = \tfrac{1}{4}l(W_1 + 2W)x^2,$$

the equation to the curve of the neutral axis; and $\xi = \frac{(W_1 + 2W)l^3}{4EI_m}$ where the value of I_m for the greatest section may be introduced.

The same result may be obtained for ξ, by observing that the effect on ξ of a distributed load $W_1 = wl$ is half of that of a collected load, hence the form of ξ with a collected load becomes with twofold loading,

$$\xi = \frac{(\frac{1}{2}W_1 + W)l^3}{2EI_m} = \frac{(W_1 + 2W)l^3}{4EI_m}.$$

Solution Number 16.—*Cantilever of uniform strength but of unsymmetrical section.*

The unsymmetric section, when rendered isotropic by design, has been already treated in Solutions Nos. 4, 8,

and 12. It may be used representatively in the same way as a symmetrical section of isotropic material; but the results will be merely approximative; and the allowance for change of position of the neutral axis must be higher with cantilevers of uniform strength than with those of uniform section, for the reason that with the former it is not horizontal.

The results of Solutions 13, 14, and 15 will under those limitations apply here; the method of applying the symmetrical section is analogous to that adopted in Solution No. 8.

For instance, with variously applied loading, and as regards horizontal stress, the value of $\frac{I}{a}$ for the typical unsymmetric section there used can be put in the form $\frac{1}{12}S\left(\frac{(n+1)(t_1^2+nt_2^2)+12nt_1^2}{t_1+t_2(n+2)+2t}\right)$ and applied in the three cases of

$$\tfrac{1}{2}wx^2=\frac{RI}{a};\ \ Wx=\frac{RI}{a};\ \ (\tfrac{1}{2}W_1+W)x=\frac{RI}{a}.$$

Hence if we adopt uniform breadth, we obtain in the first case an equation to the curve of depths that gives a parabola with its vertex at the free end of the cantilever; in the second case the line of depths is straight, so also in the third case.

Correspondingly also if we adopt uniform depth; we can then put $\frac{I}{a}$ in the form $\frac{1}{12}y.\frac{n+1}{k}(t_1^2+nt_2^2)t_1+ynkt_1$; substitute it in the general equations, and obtain the curves of breadths.

2nd and 3rd. The conditions of safety under vertical stress, and the conditions of stiffness may be taken from Solutions 13, 14, and 15, remembering the necessary limitations which cause them to be merely approximative.

FIXED GIRDERS.

Solution Number 17.—Fixed girder or beam of uniform and of symmetrical section in isotropic material, under equally distributed load.

As the complete series of stresses on a fixed girder can only be determined through employment of the elastic curve, aided by a knowledge of the stresses on a similar free girder, the first requirement is evidently to obtain the stresses wanting.

Let l be the whole length of the fixed girder,
- w the load per unit of length of l,
- E the elasticity of the material,
- I the moment of inertia of the section,
- a the extreme axial distance in the section,
- R the safe resistance of the material,
- x and y co-ordinates from an origin at the nearest abutment,
- H_x the horizontal stress at any abscissa x.

Now the property of fixture introduces two conditions, first, that the girder is held rigidly horizontal at the fixed points, so that the tangent to the curve is there zero, or $\partial_x y=0$; secondly a moment or force of fixture is introduced whose effect as regards horizontal stress we will denote by B.

Hence $$H_x=\tfrac{1}{2}wx(l-x)-B;$$

also as $$H_x=M=\frac{1}{\rho}EI=-EI.\partial^2_x y;$$

hence $$EI(-\partial^2_x y)=\tfrac{1}{2}wx(l-x)-B;$$

integrating, and noticing that when $x=0$, $\partial_x y=0$;

$$EI.\partial_x y=\tfrac{1}{6}wx^3-\tfrac{1}{4}wlx^2+Bx;$$

and as already at the limit $\delta_x y=0$ when $x=l$,

hence $\quad J=\frac{1}{12}wl^2$ and $H_x=-\frac{1}{2}w(x^2-lx+\frac{1}{6}l^2)$

is a general expression.

Now as the fixed girder may presumably consist of three portions, a cantilever at each end and a free girder in the middle, giving two points of contrary curvature, the position of these can now be determined.

At any such point $H_x=0$.

$$\therefore x^2-lx+\tfrac{1}{6}l^2=0 \quad \text{and } x=\frac{l}{2}\mp\frac{l}{\sqrt{12}}=0{\cdot}2113l;$$

when measured from either abutment.

Let l_1 be the length of the supported or free portion; then $l_1=l\times0{\cdot}5774=l/\sqrt{3}$.

We can now proceed to find:

1. The condition of safety under horizontal stress.

The weakest position will evidently be at midspan; hence we may deal with the free portion of girder and obtain (see Free Girders, Solution 1)

$$w=\frac{8RI}{al_1^2}=\frac{8RI}{al^2\times(0{\cdot}5774)^2}=\frac{24RI}{al^2}.$$

By which the strength of a fixed girder is thrice that of a free girder under these conditions.

2. Condition of stiffness.

Keeping the origin as before at the abutment,

$$EI(-\delta_x^2 y)=H_x=-\tfrac{1}{12}w(6x^2-6lx+l^2);$$

integrating between x and $\frac{1}{2}l$; and observing that at the latter limit $\delta_x y=0$, as y has then a maximum;

$$EI.\delta_x y=\tfrac{1}{12}w(2x^3-3lx^2+l^2x);$$

integrating between x and 0, observing that when $x=0$, $y=0$,

$EIy=\frac{1}{24}w(x^4-2lx^3+l^2x^2)$; the equation to the curve of the neutral axis.

Also ξ the flexure $=y$, when $x=\frac{1}{2}l$;

$$\therefore \xi=\frac{wl^4}{384EI};$$

by which the flexure of a fixed girder is one-fifth of that of a free girder under similar conditions.

Fixture at only one end.—In this case there will be only one point of contrary curvature, one cantilever portion, and the rest as free girder; the solution can then be similarly worked out as a special case of the above.

Solution Number 18.—*Fixed girder of uniform and of symmetrical section in isotropic material, under a load collected at midspan.*

Adopting the same method as in the last solution, we have

$$H_x=\tfrac{1}{2}Wx-B; \quad M=EI(-\partial^2_x y); \text{ and } H_x=M.$$

Hence $$EI(-\partial^2_x y)=\tfrac{1}{2}Wx-B;$$

integrating between limits x and $\frac{1}{2}l$, we have

$$EI.\partial_x y=\tfrac{1}{2}W(\tfrac{1}{2}x^2-\tfrac{1}{8}l^2)+B(x-\tfrac{1}{2}l);$$

but on account of fixity $\partial_x y=0$, when $x=0$; hence

$$B=\tfrac{1}{8}Wl; \text{ and } H_x=-\tfrac{1}{2}W(x-\tfrac{1}{4}l).$$

At the point of contrary flexure $H_x=0$; hence then $x=\frac{1}{4}l$ gives its position; also the length of free girder portion is $l_1=\frac{1}{2}l$.

Proceeding now to find:

1. The condition of safety under horizontal stress:

The weakest position for a section is at midspan, which is also at middle of the free portion.

Hence $$W=\frac{4RI}{al_1}=\frac{8RI}{al};$$

by which the strength of a fixed girder is twice that of a free girder under these same conditions.

2. Condition of stiffness.

Keeping the origin as before at the abutment,

$$EI(-\partial^2_x y)=H_x=-\tfrac{1}{2}W(x-\tfrac{1}{4}l);$$

integrating between the limits x and $\frac{1}{2}l$, and observing that at the latter limit $\partial_x y=0$, as $y=0$,

$$EI(-\partial_x y)=-\tfrac{1}{4}W(x^2-\tfrac{1}{2}lx)=+\tfrac{1}{4}W(-x^2+\tfrac{1}{2}lx);$$

integrating between the limits x and 0,

$$EIy=-\tfrac{1}{4}W(\tfrac{1}{3}x^3+\tfrac{1}{4}lx^2)$$

the equation to the curve of the neutral axis.

Also ξ the flexure$=y$, when $x=\frac{1}{2}l$; $\therefore\ \xi=\dfrac{Wl^3}{192EI}$.

By which the flexure of a fixed girder is one-fourth of that of a free girder under these conditions.

Solution Number 19.—*Fixed girder of uniform strength, and of symmetrical section of isotropic material, under an equally distributed load.*

Let the girder have uniform depth, and let its breadth vary so as to secure uniformity of strength under horizontal stress.

Under this condition the curvature will be constant, though varying in sign at either side of the point of contrary curvature. To determine the position of this point, take it for origin, measuring abscissæ, $+x$ and $-x$ on either side of it along the free girder portion and the cantilever portion, in such a way that corresponding equal values of

x are taken; we can then compare the horizontal forces, which under constant curvature will be equal. In the free girder the one is $\frac{1}{2}wx(l_1-x)$, when l_1 is the length of the free girder portion, in the cantilever the other $\frac{1}{2}wx^2$, and at the neutral point their difference is zero.

$$\therefore \tfrac{1}{2}wx(l_1-x)-\tfrac{1}{2}wx^2=0\,;\ \text{or } x=\tfrac{1}{2}l_1\,;$$

but with extreme values of x, $\frac{1}{2}l_1+x=\frac{1}{2}l$;

$$\therefore x=\tfrac{1}{4}l, \text{ and } l_1=\tfrac{1}{2}l.$$

Also, under these conditions, the general expression for the horizontal forces in the free girder portion before used, $\frac{1}{2}wx(l_1-x)$, will give the extreme value at midspan.

Hence removing the origin now to midspan, the above becomes $\frac{1}{2}w(\frac{1}{4}l_1^2-x^2)$, and when $x=0$, it is $\frac{1}{8}wl_1^2=\frac{1}{32}wl^2$.

But if the whole of the girder were free, the extreme value of the horizontal force at midspan would be $\frac{1}{8}wl^2$. Therefore the result of fixity is to introduce a horizontal force $=\frac{1}{8}wl^2-\frac{1}{8}wl_1^2=\frac{3wl^2}{32}=B$; and as the moment is constant everywhere, we have $H_x=\frac{1}{2}w(\frac{1}{4}l^2-x^2)-\frac{3wl^2}{32}$.

1. Condition of safety under horizontal stress.

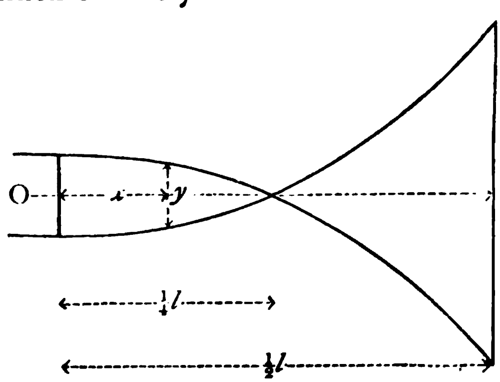


FIGURE 14.—Curve of breadths.

At midspan the equation of stress and strain will be

$H_x = M$ when $x=0$ in the last given expression; or

$$\tfrac{1}{32}wl^2 = \frac{RI_m}{a},$$

and the breadth here will be b_m, a function of I_m thus determined for midspan section.

At the point of contrary curvature, where $x=\frac{1}{4}l$, $H_x=0$, and the breadth becomes zero, if we neglect vertical stresses.

At the abutment where $x=\frac{1}{2}l$, $H_x=\frac{3wl^2}{32}=\frac{RI}{2}$, and a fixed value of breadth results.

Any intermediate values of b, the breadth, can be found through the general equation

$$\tfrac{1}{2}w(\tfrac{1}{4}l^2-x^2)-\frac{3wl^2}{32}=\frac{RI}{a},$$

where I and its function y may be deduced. The curve of breadths is parabolic as shown in the figure.

2. Condition of stiffness.

The moment of resistance to deflexion is everywhere constant, and is the greatest value, $\frac{1}{32}wl^2$, before given; the curvature is constant also

$$\therefore EI_m.\delta^2_x y = \tfrac{1}{32}wl^2; \text{ and } EI_m y = {}_0^x\!\int {}_0^x\!\int_x \tfrac{1}{32}wl^2 = \tfrac{1}{64}wl^2x^2,$$

the equation to the curve of the neutral axis.

Also the flexure $\xi = y$, when $x=\frac{1}{2}l$; $\therefore \xi = \dfrac{wl^4}{256EI_m}$.

The case of uniform breadth is said to be indeterminate in any similar complete form.

Solution Number 20.—*Fixed girder of uniform strength, and of symmetrical section of isotropic material, under a load collected at midspan.*

Let the girder have uniform depth, and let its breadth vary so as to secure uniformity of strength under horizontal stress.

Under this condition the curvature will be constant, as in the preceding solution; also the points of contrary flexure will be similarly situated, each cantilever portion having its length $=\frac{1}{4}l$; where l=whole span.

The horizontal forces will exactly correspond to those for a fixed girder of uniform section similarly loaded (see Solution 18); that is, using an origin at the abutment, $H_x=-\frac{1}{2}Wx+\frac{1}{8}Wl.$

1. Condition of safety under horizontal stress.

At midspan $H_x=-\frac{1}{8}Wl=\frac{RI}{a}$; or $W=\frac{8RI}{al}$; and this condition will hold for every section to secure uniform strength; that is,

$W=\frac{8RI_m}{al}$; I_m being the moment of inertia at the mid-span section.

2. Condition of stiffness.

The curvature is dependent on the greatest horizontal stress; this occurs when $x=\frac{1}{2}l$; and then $H_x=\frac{1}{8}Wl.$

Taking now a new origin at midspan, and treating the curvature as constant, $EI_my={}^x_0\!\!\int{}^x_0\!\!\int_x \frac{1}{8}Wl=\frac{1}{16}Wlx^2$;

and $\xi=y$, when $x=\frac{1}{2}l$; $\therefore\ \xi=\frac{1}{64}\cdot\frac{Wl^3}{EI_m}$.

The case of uniform depth is said to be indeterminate, in any similar complete form.

Fixed girders are comparatively rarely used, on account of the high temperature stress to which they are liable.

CONTINUOUS GIRDERS.

The material ordinarily employed in continuous beams or girders is nearly isotropic, as timber and wrought iron.

Continuous girders are horizontal, that is the support-

[illegible] and should be maintained [illegible]

Continuous girders [illegible] are necessarily free to obtain the advantages of continuity, and to avoid high temperature stresses [illegible] the two ends might be fixed.

Continuous beams or girders are of uniform section, or of approximately uniform section, as some reduction may be applied in parts. The principle of uniform strength cannot be theoretically adhered to throughout a solution, though it may be partially employed to modify results.

Assuming the above principles, it will also be taken for granted that the safe resistances to horizontal and to vertical stress can [illegible] be applied at the points of maximum stress of either sort when these are found.

These solutions will therefore be confined to determining

(1) The amount and position of the maximum horizontal stress occurring in each span.

(2) The amount and position of the maximum vertical stress.

(3) The deflexions.

Cases of continuous girders may be solved when there is any number of spans of different lengths and with any varying mode of loading, under the before-mentioned conditions. But the more commonly required cases are those with loading uniformly distributed in each span. These may again be divided into classes thus:

(1) The loading uniform over all the spans throughout.

(2) The loading symmetrical on spans correlatively situated with regard to the middle of the whole girder.

(3) The loading different in all the spans, although distributed equally in each single span.

(4) The spans equal throughout.

(5) The spans symmetric in length correlatively to the middle point of the whole girder.

(6) All the spans having unequal lengths.

Also there are various combinations of these distinctive classes.

There are two *modes* of solution with continuous girders; the one through the Theorem of Three Moments and general treatment by partial reactions; the other by detached treatment independently and through total reactions.

Application of the General Theorem.

The Theorem of Three Moments applied to any two contiguous spans of a continuous horizontal girder in its most general form is ascribed to Bresse; its most useful applications are expressed under the following heads:

1. A continuous girder consisting of n spans and having $n+1$ supports, when treated in pairs of contiguous spans, for which the relations of moments are given by the theorem, will then afford $n-1$ equations. Hence if any two of the reaction-couples at supports be known, and this is usually the case with end supports, the whole of the remaining reaction-couples can be found by this process. Conditions of symmetry would further reduce the number of unknown moments.

2. In the equation of curvature or inclination for any one span of a contiguous pair, the inclination of the curve may and generally does differ at one end from that at the other, even after allowing for change of sign. Thus if $EI\partial_x^2 y = H_x$ in any span, we may have $EI.(\partial_x y - \tan\beta) = {}_0^x\!\int_x H_x$; and $\tan\beta$ is required. The equation for the contiguous span at the intermediate point of support affords the value of $EI.\tan\beta$.

3. The equation to the curve in each span of any contiguous pair, whence deflexions may be obtained, can be

represented in a convenient form, independently of other spans.

The general principle on which the three formulæ used in these applications are based may be thus briefly expressed:

General Theorem.—The general expression for the elastic curve in either span of a contiguous pair (see figure)

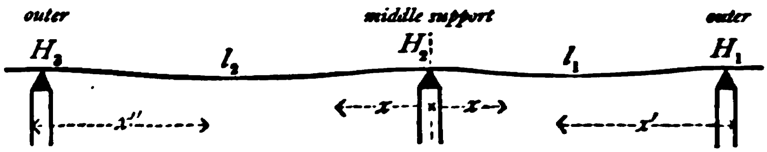


FIGURE 15.—Any two contiguous spans.

is $EI\partial_x^2 y = H_x$; as frequently used with detached spans already; but here x, the abscissa, is measured invariably outwards from O, the origin at the middle support.

Whence by integration in each separate span

$$EI(\partial_x y - \tan\beta) = {}_0^x\!\int_x H; \text{ and } EI(\partial_x y + \tan\beta) = {}_0^x\!\int_x H;$$

where H is different in each span, but $\tan\beta$ is common to both at the intermediate point of support; also, when $x=0$, $\partial_x y - \tan\beta = 0$; or $\partial_x y + \tan\beta = 0$.

Integrating again, and observing that in both cases, when $x=0$, or when $x=l$, $y=0$, as the girder is strictly horizontal, we have

$$EI(-l_1 \tan\beta) = {}_0^{l_1}\!\int {}_0^x\!\int_x H; \text{ and } EI(+l_2 . \tan\beta) = {}_0^{l_2}\!\int {}_0^x\!\int_x H;$$

general expressions in which the two separate general values of H, one for each span, must be replaced by respectively assignable quantities.

Taking the first span only, and retaining the origin at O,

$${}_0^{l_1}\!\int {}_0^x\!\int_x H = l_1 \, {}_0^{l_1}\!\int_x H - {}_0^{l_1}\!\int_x x . \partial_x \, {}_0^x\!\int_x H = {}_0^{l_1}\!\int_x (l_1 - x) . H.$$

Now by transfer to a new origin at the outer support (at H_1) and by evaluating the general expression H at any abscissa x (see Stresses on Continuous Girders, Chapter II. Part I. pages 26 and 27).

$$\int_0^{l_1}{}_x (l-x).H = \int_0^{l_1}{}_{x'} x'\left\{H_1 + \frac{x'}{l_1}(H_2 - H_1) + H_d\right\};$$

where H_1 is the moment of reaction at the outer support,
„ H_2 „ „ „ middle support,
„ H_d is the general moment of horizontal force for the same span, when discontinuous.

But this expression may be reduced to

$$\tfrac{1}{2}l_1^2 H_1 + \tfrac{1}{3}l_1^2(H_2 - H_1) + \int_0^{l_1}{}_{x'} x' H_d$$
$$= \tfrac{1}{6}l_1^2 H_1 + \tfrac{1}{3}l_1^2 H_2 + {}_0^{l_1}\left\{\tfrac{1}{2}x'^2 . H_d\right\} - \tfrac{1}{2}\int_0^{l_1}{}_{x'} x'^2 . \frac{\partial H_d}{\partial x'}.$$

It may also be further reduced; for as H_d vanishes at both limits, $x'=0$, and $x'=l_1$; and as $V_d = \frac{\partial H_d}{\partial x'}$, the general expression for shearing stress when the span is discontinuous;

putting $K_1 = \int_0^{l_1}{}_{x'} V_d \frac{x'^2}{l_1}$;

the above becomes $\tfrac{1}{6}l_1^2 H_1 - \tfrac{1}{3}l_1^2 H_2 - \tfrac{1}{2}\int_0^{l_1}{}_{x'} . x'^2 . V_d$,

whence $\qquad EI(-\tan\beta) = \tfrac{1}{6}l_1 H_1 - \tfrac{1}{3}l_1 H_2 - \tfrac{1}{2}K_1. \quad . \quad . \quad . \quad$ (I.)

Adopting a corresponding process for the other span l_2, and transferring the origin to the outer support at H_3, using x'' as the new abscissa and putting similarly $K_2 = \int_0^{l_2}{}_{x''} V_d . \frac{x''^2}{l_2}$, we get also

$$EI(+\tan\beta) = \tfrac{1}{6}l_2 H_3 + \tfrac{1}{3}l_2 H_2 - \tfrac{1}{2}K_2. \quad . \quad . \quad . \quad \text{(II.)}$$

And from equations I. and II.

$$H_1 l_1 + 2H_2(l_1 + l_2) + H_3 l_2 = 3(K_1 + K_2) \quad . \quad . \quad \text{(III.)}$$

the required equation of three moments, that can be applied in Application No. 1, before given. In using any of these three equations it must be remembered that K_1 and K_2 are expressions referred to different origins.

For the purpose of Application No. 2 before given, that is to find $\tan\beta$ in any case, either of the equations (I.) or (II.) may be used as required.

For the purpose of Application No. 3 before mentioned, that is to use the equation to the curve for either of the two spans; taking the origin again at the middle support, a process similar to the former part of the preceding will yield the two following equations Nos. IV. and V.:

$$EI.l_1 y = \tfrac{1}{6}H_1(x^3 - l_1^2 x) + \tfrac{1}{6}H_2(3l_1 x^2 - x^3 - 2l_1^2 x)$$
$$+ \tfrac{1}{2}l_1 x.K_1 + l_1.{}_0^x\!\int_x {}_0^x\!\int_x H_d; \quad . \quad . \quad . \quad \text{(IV.)}$$

$$EI.l_2 y = \tfrac{1}{6}H_3(x^3 - l_2^2 x) + \tfrac{1}{6}H_2(3l_2 x^2 - x^3 - 2l_2^2 x)$$
$$+ \tfrac{1}{2}l_2 x.K_2 + l_2\,{}_0^x\!\int_x {}_0^x\!\int_x H_d; \quad . \quad . \quad . \quad . \quad \text{(V.)}$$

The following tables of values of K and of ${}_0^x\!\int_x {}_0^x\!\int_x H_d$ taken from Cunningham's 'Applied Mechanics,' 1876, pages 357 and 362, enable these results (expressed in the form adopted by him) to be applied to the curves of any spans; so that the curves may be plotted direct from them without need of further integration, after substituting the values of H_1, H_2, H_3, &c., in numerical quantities.

The before-mentioned Applications will be made in the general solutions following, whenever required, without adopting any other form.

Table of Values of K

for use in Equations I., II., III., IV., V., before given;

where $$K = \int_0^l \frac{x^2}{l} . V_d . \partial x; \text{ and } V_d = \partial_x H_d;$$

V_d and H_d being values under the assumption of discontinuity.

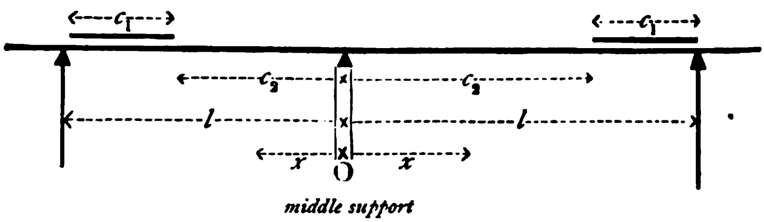


FIGURE 16.

Loading	With origin at outer support for c_1	With origin at middle support for c_2
Uniform load $(-w)$:		
over the whole span l.	$-\frac{1}{12}wl^3$;	
over a length c_1 . .	$\frac{1}{12}\cdot\frac{w}{l}\cdot c_1^2(c_1^2-2l^2)$;	$\frac{1}{12}\frac{w}{l}(l-c_2)^2.(c_2^2-2lc_2-l^2)$;
over a length c_2 . .	$-\frac{1}{12}\cdot\frac{w}{l}(l^2-c_1^2)^2$;	$-\frac{1}{12}\frac{w}{l}\cdot c_2^2(2l-c_2)^2$;
Detached load $(-W)$:		
at midspan, $\frac{1}{2}l$ . .	$-\frac{1}{8}Wl^2$	
at distance c_1 . .	$\frac{1}{3}W.\frac{c_1}{l}(c_1^2-l^2)$	$\frac{1}{3}W.\frac{c_2}{l}(c_2-l)(2l-c_2)$;
Equal loads each $(-W)$:		
Two loads, each distant c from the nearest end of the span .	$Wc_1(c_1-l)$;	
$n-1$ loads, so placed as to divide l into n parts . . .	$-\frac{W}{12n}.l^2(n^2-1)$;	

Note.—All these values of K are negative, as either c_1 or c_2 is less than l. Also the sum of values of K for separate or partial loads is the value of K for the total load. See figure for c_1 and c_2, &c.

Table of Values of ${}_0\int^x {}_0\int^x H_d$.

for use in Equations (IV.) and (V.); for the curve in either span, when the origin is at the middle support (see figure 16).

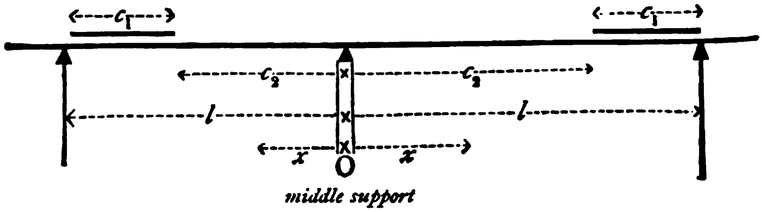


FIGURE 16.

Loading	Limit of x	Value of ${}_0\int_x^2 H_d$
Uniform load $(-w)$:		
over whole span l .	anywhere	$\frac{1}{12}w(lx^3-\frac{1}{2}x^4)$;
over the length c_1 .	$x<c_2$	$\frac{1}{12}w.c_1^2.\frac{x^3}{l}$;
do. .	$x>c_2$	$\frac{1}{24}wc_1^2\{2x(l+2x)-(l+c_2^2)\}$ $+\frac{1}{24}\frac{w}{l}(l-x)^3.(l^2+lx-2c_2^2)$;
over the length c_2 .	$x<c_2$	$\frac{1}{24}wx^3\left\{-x+\frac{l^2-c_1^2}{l}\right\}$;
do.	$x>c_2$	$\frac{1}{24}w\frac{c_1^2}{l}\left\{6lx-2x^3-4lc_2x+lc_2^2\right\}$;
Detached load $(-W)$:		
at midspan $\frac{1}{2}l$ . .	$x<\frac{1}{2}l$	$\frac{1}{12}Wx^3$;
	$x>\frac{1}{2}l$	$\frac{1}{48}W(12lx^2-4x^3-6l^2x+l^3)$;
at a distance c_1 from outer support . .	$x<c_2$	$\frac{1}{6}Wc_1.\frac{x^3}{l}$;
	$x>c_2$	$\frac{1}{6}Wc_2\left\{3x^2-\frac{x^3}{l}-3c_2x+c_2^2\right\}$;
Equal loads (each $-$ W):		
each distant c_2 from the nearest end of the span, so that $c_1=l-c_2$	$x<c_2$	$\frac{1}{6}Wx^3$;
	$x>c_2$; $x<c_1$	$\frac{1}{2}Wc_2(x^2-c_2x+\frac{1}{3}c_2^2)$;
	$x>c_1$	$\frac{1}{6}W\{(l-x)^3+3c_1c_2(2x-l)\}$;

GENERAL SOLUTION.

Solution Number 21.—*Continuous girder of any number* (n) *of equal spans, having the same equally distributed load-intensity over the whole.*

The general theorem of moments of horizontal forces on a continuous girder holds with regard to any two contiguous spans l_1 and l_2 in it; and is expressed thus in (III.)

$$H_1 l_1 + 2H_2(l_1+l_2) + H_3 l_2 = 3(K_1+K_2)$$

where H_1, H_2, H_3 are the horizontal forces at the three supports.

K_1 and K_2 are quantities corresponding in the two spans; generally $\qquad K_1 = \int_0^{l_1} (l_1-x)^2 . V_d . \delta(l_1-x)$

where V_d is the vertical force under the assumption of discontinuity; and x is measured from an origin at the middle support.

Now making use of the Table of Values of K and with equally distributed load on each span, the above becomes $H_1 l_1 + 2H_2(l_1+l_2) + H_3 l_2 = -\frac{1}{4}w_1 l_1^3 - \frac{1}{4}w_2 l_2^3$; and with equal spans throughout, it further becomes

$$H_1 + 4H_2 + H_3 = -\tfrac{1}{4}l^2(w_1+w_2);$$

and with equal loading throughout, $w_1 = w_2$ everywhere, it becomes $\qquad H_1 + 4H_2 + H_3 = -\frac{1}{2}wl^2$;

the form suited to this particular general solution. Applying it to n spans, we have first independently

$$H_1 = H_{n+1} \qquad = 0 \text{ at the free}$$

ends, and afterwards

$$\begin{aligned}
4H_2 + H_3 = H_{n-1} + 4H_n &= -\tfrac{1}{2}wl^2 \\
H_2 + 4H_3 + H_4 = H_{n-2} + 4H_{n-1} + H_n &= -\tfrac{1}{2}wl^2 \\
H_3 + 4H_4 + H_5 = H_{n-3} + 4H_{n-2} + H_{n-1} &= -\tfrac{1}{2}wl^2 \\
\&c., \qquad \&c. \qquad &= \quad \&c.
\end{aligned}$$

and can have as many equations as there are quantities; while the condition of symmetry on either side of the middle, giving $H_2 = H_n$; $H_3 = H_{n-1}$; $H_4 = H_{n-2}$, &c. = &c.; further simplifies them. So that the whole of the values from H_1 to H_{n+1} can be reduced. (See Table following.)

Now let us distinguish reactions, &c., to the right and to the left of each span by respectively single and double dashes, and let the subscript now designate the span.

Let A'_p and A''_p be the reactions to the right and left of the p^{th} span, when supposed discontinuous,

B'_p nd B''_p be the additional reactions there introduced by continuity.

But generally $H'' = H' + B'.l$; and $H' = H'' + B''l$;

$$\therefore\ B'_p = \frac{H_{p+1} - H_p}{l_p}; \text{ and } B''_p = \frac{H_p - H_{p+1}}{l_p};$$

and in values at the two ends

$$B'_1 = \frac{H_2 - H_1}{l_1}; \text{ and } B''_n = \frac{H_n - H_{n+1}}{l_n};$$

We can thus obtain the whole series of values of B from those of H. Adding to them the simple values of A, we can tabulate the series $A'_p + B'_p$; $A''_p + B''_p$; for any p^{th} span.

If we now distinguish abscissæ measured in any span from an origin at the right end of it by a single dash, as x', and from an origin at the left end of it by a double dash as x'',

and while H_p remains the value of H at the p^{th} support; let H_x be the value of H at any abscissa x in any span. Then in the p^{th} span we have

$$H_x = H_p + (A'_p + B'_p)x' - \tfrac{1}{2}wx'^2 = H_{p+1} + (A''_p + B''_p)x'' - \tfrac{1}{2}wx''^2;$$

the general expression for horizontal force; where for the two end spans, the first terms vanish, when $p=1$ or $p=n$.

Now let V'_p and V''_p be the vertical forces to the right and left of any span, the p^{th}; then as generally at supports

$$V'=A'+B'\text{ ; and }V''=-(A''+B''),$$

we have, if V_x be the value of V at any abscissa x in any span,

$$V_x=A'_p+B'_p-wx'=-(A''_p+B''_p)-wx'',$$

the general expression for vertical force in any span.

To obtain maximum stress values.

The positive maximum values of H_x, which we denote by H_m, will necessarily occur at the same position where $V_x=0$, whence the abscissa for the section is

$$x'=\frac{1}{w}(A'_p+B'_p);\text{ or }x''=\frac{1}{w}(A''_p+B''_p)$$

and by substitution in the value of H_x, we have in any p^{th} span

$$H_m=H_p+\frac{1}{2w}(A'_p+B'_p)^2=H_{p+1}+\frac{1}{2w}(A''_p+B''_p)^2;$$

the negative maximum values of H are at all the supports H_2, H_3, &c., except the last. Also as $V=\partial_x H$, the positive maximum values of V will also occur at the same places, or $V'_m=A'+B'$; and $V''_m=-(A''+B'')$.

The condition of $H_x=0$ gives the points of inflexion, whence from the value of H_x their abscissæ are

$$x'=\frac{1}{w}\left(A'_p+B'_p\pm\sqrt{(A'_p+B'_p)^2+2w.H_p}\right)$$

$$x''=\frac{1}{w}\left(A''_p+B''_p\pm\sqrt{(A''_p+B''_p)^2+2w.H_{p+1}}\right)$$

which combine only in the end spans to form one point.

The following table exemplifies the general solution of the stresses; giving values for cases from two to six spans. Hence values of H_m in any span, and of V_m at any support are determined, for which sections of flanges and of web affording safe resistance can be designed and calculated direct, according to the methods adopted in foregoing solutions of separate girders.

Stresses at Supports; for Equal Spans, and Equable Loading.

Horizontal Stress	Two Spans	Three Spans	Four Spans	Five Spans	Six Spans
H_1	o	o	o	o	o
H_2	$-\frac{1}{8}wl^2$	$-\frac{1}{10}wl^2$	$-\frac{3}{28}wl^2$	$-\frac{2}{19}wl^2$	$-\frac{11}{104}wl^2$
H_3	o	$-\frac{1}{10}wl^2$	$-\frac{1}{14}wl^2$	$-\frac{3}{38}wl^2$	$-\frac{1}{13}wl^2$
H_4	. . .	o	$-\frac{3}{28}wl^2$	$-\frac{3}{38}wl^2$	$-\frac{9}{104}wl^2$
H_5	. . .	. . .	o	$-\frac{2}{19}wl^2$	$-\frac{1}{13}wl^2$
H_6	. . .	. . .	. . .	o	$-\frac{11}{104}wl^2$
H_7	. . .	. . .	. . .	. . .	o
Reactions.					
$A'_1+B'_1$	$\frac{3}{8}wl$	$\frac{2}{5}wl$	$\frac{11}{28}wl$	$\frac{15}{38}wl$	$\frac{41}{104}wl$
$A''_1+B''_1$	$\frac{5}{8}wl$	$\frac{3}{5}wl$	$\frac{17}{28}wl$	$\frac{23}{38}wl$	$\frac{31}{52}wl$
$A'_2+B'_2$	$\frac{5}{8}wl$	$\frac{1}{2}wl$	$\frac{15}{28}wl$	$\frac{10}{19}wl$	$\frac{55}{104}wl$
$A''_2+B''_2$	$\frac{3}{8}wl$	$\frac{1}{2}wl$	$\frac{13}{28}wl$	$\frac{9}{19}wl$	$\frac{49}{104}wl$
$A'_3+B'_3$	. . .	$\frac{3}{5}wl$	$\frac{13}{28}wl$	$\frac{1}{2}wl$	$\frac{51}{104}wl$
$A''_3+B''_3$	. . .	$\frac{2}{5}wl$	$\frac{15}{28}wl$	$\frac{1}{2}wl$	$\frac{53}{104}wl$
$A'_4+B'_4$	. . .	. . .	$\frac{17}{28}wl$	$\frac{9}{19}wl$	$\frac{53}{104}wl$
$A''_4+B''_4$	. . .	. . .	$\frac{11}{28}wl$	$\frac{10}{19}wl$	$\frac{51}{104}wl$
$A'_5+B'_5$	. . .	. . .	. . .	$\frac{23}{38}wl$	$\frac{49}{104}wl$
$A''_5+B''_5$	. . .	. . .	. . .	$\frac{15}{38}wl$	$\frac{55}{104}wl$
$A'_6+B'_6$	. . .	. . .	. . .	. . .	$\frac{31}{52}wl$
$A''_6+B''_6$	. . .	. . .	. . .	. . .	$\frac{41}{104}wl$

Greatest Horizontal Stress in each Span.

Maximum Stress	Two Spans	Three Spans	Four Spans	Five Spans	Six Spans
H_{m1}	$\frac{9}{128}wl^2$	$\frac{2}{25}wl^2$	$\frac{121}{1568}wl^2$	$\frac{225}{2888}wl^2$	$\frac{420}{5408}wl^2$
H_{m2}	. . .	$\frac{1}{40}wl^2$	$\frac{57}{1568}wl^2$	$\frac{112}{361}wl^2$	$\frac{184}{5408}wl^2$
H_{m3}	. . .	$\frac{2}{25}wl^2$	$\frac{57}{1568}wl^2$	$\frac{7}{152}wl^2$	$\frac{234}{5408}wl^2$
H_{m4}	. . .	. . .	$\frac{121}{1568}wl^2$	$\frac{12}{361}wl^2$	$\frac{934}{5408}wl^2$
H_{m5}	. . .	. . .	. . .	$\frac{225}{2888}wl^2$	$\frac{184}{5408}wl^2$
H_{m6}	. . .	. . .	. . .	. . .	$\frac{420}{5408}wl^2$
Abscissæ for H_m					
to m_1 from A_1	$\frac{3}{8}l$	$\frac{2}{5}l$	$\frac{11}{28}l$	$\frac{15}{38}l$	$\frac{41}{104}l$
to m_2 „ A_2	$\frac{5}{8}l$	$\frac{1}{2}l$	$\frac{15}{28}l$	$\frac{10}{19}l$	$\frac{55}{104}l$
to m_3 „ A_3	. . .	$\frac{3}{5}l$	$\frac{13}{28}l$	$\frac{1}{2}l$	$\frac{51}{104}l$
to m_4 „ A_4	. . .	. . .	$\frac{17}{28}l$	$\frac{9}{19}l$	$\frac{53}{104}l$
to m_5 „ A_5	. . .	. . .	. . .	$\frac{23}{38}l$	$\frac{49}{104}l$
to m_6 „ A_6	. . .	. . .	. . .	. . .	$\frac{63}{104}l$

Deflexion of girders having equal spans and loading uniform throughout.

First, with a two-span girder.

The points of inflexion are obtained by putting $H_x=0$ in any span;

here $H_x=H_1+(A'_1+B'_1)x'-\frac{1}{2}wx'^2$; (see Table),

$$=0+\tfrac{3}{8}wlx-\tfrac{1}{2}wx^2=0,$$

whence $x=\frac{3}{4}l$ gives the point of inflexion.

Also for deflexion; $-EI.\,\delta^2_x y=\frac{3}{8}wlx-\frac{1}{2}wx^2$; integrating between x and l, observing that when

$x=l$, $\delta_x y-\tan\beta=0$; whence $C=+\frac{1}{48}wl^3$;

$EI(\delta_x y-\tan\beta)=\frac{1}{6}wx^3-\frac{3}{16}wlx^2+C.$

... in the second span at the same support,

$$- \ldots = \ldots - \tfrac{1}{2}K_2 = 0 - \tfrac{1}{24}wl^3 + \tfrac{1}{24}wl^3 = 0.$$

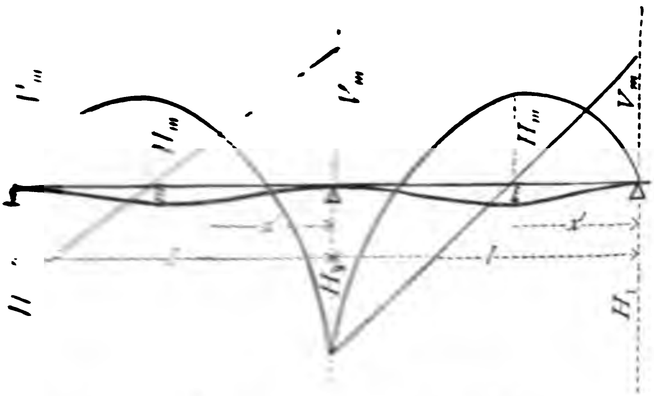


FIGURE 17.—Two-span continuous girder of equal spans.

... the former equation between x and 0,

$$\ldots = \ldots wx^4 - \tfrac{1}{16}wlx^3 = \tfrac{1}{48}wl^3x.$$

To obtain an abscissa x to point of ξ the flexure, put ... $x^3 - \tfrac{9}{8}lx^2 + \tfrac{1}{8}l^3 = 0$; or $\frac{x^3}{l^3} - \frac{9}{8}\frac{x^2}{l^2} + \frac{1}{8} = 0$; solving this $x = 0.4215l$ or $0.5785l$; and by substitution in the equation to the curve, we obtain $\xi = 0.00545\,\frac{wl^4}{EI}$.

The second span corresponds in converse order.

The values of H_m and of V_m and their positions have been already indicated, and given in the Table.

Next as to deflexion with three spans.

The points of inflexion in any span are first obtained by putting $H_x = 0$.

For instance, in the first span of a 3-span girder,

$$H_x = H_1 + (A'_1 + B'_1)x' - \tfrac{1}{2}wx'^2 \ldots$$

$$= 0 + \tfrac{2}{5}wlx' - \tfrac{1}{2}wx'^2 \ldots$$

whence $x' = \tfrac{4}{5}l$ give...

Now, to obtain the deflexions in the same case,
$EI.\partial^2_x y=H_x=\frac{2}{5}wlx-\frac{1}{2}wx^2$,
integrating between x and 0, and observing that when $x=l$, $y=0$, and $\partial_x y-\tan\beta=0$, β being the inclination to horizontality of the curve at l;

$EI(\partial_x y-\tan\beta)=\frac{1}{5}wlx^2-\frac{1}{6}wx^3+C$;

where $C=-\frac{1}{30}wl^3$.

Integrating between x and 0, and noticing [illegible]
$x=0$, $y=0$;

$EIy=\frac{1}{15}wlx^3-\frac{1}{24}wx^4-\frac{1}{30}wl^3x+x.EI.\tan\beta$.

To evaluate $\tan\beta$, which is common [illegible] second spans at the intermediate support [illegible] second span, where w and l are the same [illegible] the equation of moments,

$$-EI.\tan\beta=\frac{1}{6}lH_3+\frac{1}{3}lH_2-\frac{1}{2}K_2$$
$$=-\frac{1}{60}wl^3-\frac{1}{30}wl^3+\frac{1}{24}\text{[illegible]}$$
$$\therefore EIy=\frac{1}{15}wlx^3-\frac{1}{24}wx^4-\text{[illegible]}$$
$$\text{or}\quad EIy=\frac{1}{15}wlx^3-\text{[illegible]}$$

which holds when $x=l$, as y [illegible] to the curve of the neutral [illegible] of finding the flexure ξ or [illegible] curve, and scale off the [illegible] Analytically the so[illegible] ${}^3)$; involves a cubic eq[illegible] to solve it, when y [illegible] $\therefore\frac{1}{5}lx^2-\frac{1}{30}l^3-\frac{1}{8}x^3=$ [illegible] $6x^2_1-5x^3_1=1$, and [illegible] repeated approxi[illegible] H_{n+1}, and [illegible] $0{\cdot}5583l$, th[illegible]

value of x in the equation to the curve, we obtain for the value of y correspondingly, $\xi = -0{\cdot}0070\,\frac{wl^4}{EI}$.

The foregoing is a typical mode of treating the elastic curve in each span singly, after having already obtained H_x the general expression for horizontal stress on it, by making use of the quantity $\tan\beta$ due to the contiguous span, for the intermediate point of support, where β is the inclination of the curve.

Also in the middle span of the 3-span girder,

$$H_x = H_2 + (A'_2 + B'_2)x - \tfrac{1}{2}wx^2 = -\tfrac{1}{10}wl^2 + \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2;$$

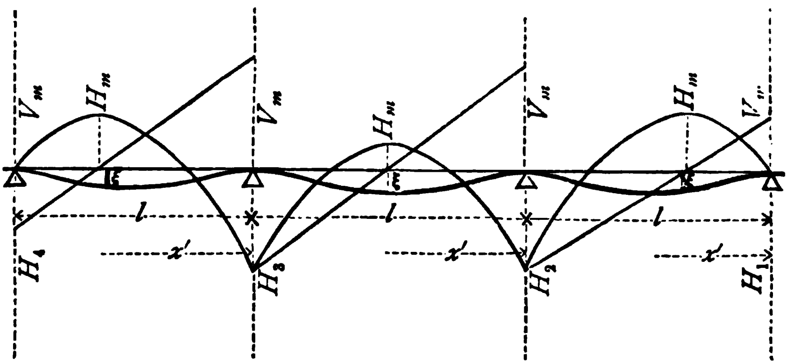


FIGURE 18. —Three-span continuous girder of equal spans.

equating this to zero, the abscissæ of the two points of inflexion are obtained from the result, $x = \tfrac{1}{2}l\left(1 \pm \frac{1}{\sqrt{5}}\right)$.

The point of flexure is from symmetry necessarily at midspan; its abscissa is $x = 0{\cdot}5l$. The required $\tan\beta$ will have a contrary sign $= -\tfrac{1}{120}wl^3$;

$$EI.\,\partial^2_x y = H_x = -\tfrac{1}{10}wl^2 + \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2;$$

$$EI(\partial_x y - \tan\beta) = -\tfrac{1}{10}wl^2x + \tfrac{1}{4}wlx^2 - \tfrac{1}{6}wx^3 + C,$$

where $C = \tfrac{1}{60}wl^3$;

$$\therefore EIy = -\tfrac{1}{20}wl^2x^2 + \tfrac{1}{12}wlx^3 - \tfrac{1}{24}wx^4 + \tfrac{1}{120}wl^3x;$$

the equation to the curve of the neutral axis,

where, if $x=\frac{1}{2}l$, $y=\xi=0{\cdot}00052\,\dfrac{wl^4}{EI}$.

The third span is conversely the same as the first. The values of H_m and V_m and their positions have been already indicated and given in the Table.

GENERAL SOLUTION.

Solution Number 22.—Continuous girder of any number of spans whose lengths are symmetrically paired with reference to the middle, but having the same equally distributed load-intensity over the whole.

The mode of solution will be analogous to that in the foregoing number, but will vary in detail. Taking the general theorem before given. (See Eq. III. on page 178.)

$$H_1l_1+2H_2(l_1+l_2)+H_3l_2=3(K_1+K_2),$$

it here becomes with equable loading and varying spans (see Table of Values of K on page 179),

$$H_1l_1+2H_2(l_1+l_2)+H_3l_2=-\tfrac{1}{4}w(l_1^3+l_2^3).$$

Applying it to n spans, we have first independently $H_1=H_{n+1}=0$ at the free ends, and then

$$2H_2(l_1+l_2)+H_3l_2=H_{n-1}l_{n-1}+2H_n(l_{n-1}+l_n)=-\tfrac{1}{4}w(l_1^3+l_2^3);$$
$$H_2l_2+2H_3(l_2+l_3)+H_4l_3=H_{n-2}l_{n-2}+2H_{n-1}(l_{n-2}+l_{n-1})+H_nl_{n-1}=-\tfrac{1}{4}w(l_2^3+l_3^3);$$
$$H_3l_3+2H_4(l_3+l_4)+H_5l_4=H_{n-3}l_{n-3}+2H_{n-2}(l_{n-3}+l_{n-2})+H_{n-1}l_{n-2}=-\tfrac{1}{4}w(l_3^3+l_4^3);$$
$$\&c.=\&c.$$

The conditions of symmetry also give

$$l_1=l_n;\ l_2=l_{n-1};\ l_3=l_{n-2};\ l_4=l_{n-3};\ \&c.$$
$$H_2=H_n;\ H_3=H_{n-1};\ H_4=H_{n-2};\ H_5=H_{n-3};\ \&c.$$

Hence we may obtain values of H from H_1 to H_{n+1}, and tabulate them, thus:

Stresses at supports for Symmetric Spans and Equable Loading

(in general terms).

Horizontal Stress	Two spans	Three spans	Four spans	Five spans
H_1	0	0	0	0
H_2	$-\frac{1}{8}wl_1^2$	$-\frac{1}{4}w\left\{\frac{l_1^3+l_2^3}{2l_1+3l_2}\right\}$	$-\frac{1}{4}w\left\{\frac{2l_1^3+l_2^3}{4l_1+3l_2}\right\}$	$-\frac{1}{4}w\left\{\frac{(2l_2+3l_3)(l_1^3+l_2^3)-(l_2^3+l_3^3)}{(2l_2+3l_3)(2l_1+2l_2)-l^2_2}\right\}$
H_3	0	$=H_2$	$-\frac{1}{4}w\frac{2l_1l_2^2+l_2^3-l_1^3}{4l_1+3l_2}$	$-\frac{H_2l_2+\frac{1}{4}w(l_2^3+l_3^3)}{2l_2+3l_3}$
H_4	. .	0	$=H_2$	$=H_3$
H_5	. .	. . .	0	$=H_2$
H_6	. .	. . .	. . .	0
Reactions				
$A'_1+B'_1$	$\frac{3}{8}wl_1$	$\frac{1}{2}wl_1+\frac{H_2}{l_1}$	$\frac{1}{2}wl_1+\frac{H_2}{l_1}$	$\frac{1}{2}wl_1+\frac{H_2}{l_1}$
$A''_1+B''_1$	$\frac{5}{8}wl_1$	$\frac{1}{2}wl_1-\frac{H_2}{l_1}$	$\frac{1}{2}wl_1-\frac{H_2}{l_1}$	$\frac{1}{2}wl_1-\frac{H_2}{l_1}$
$A'_2+B'_2$	$\frac{5}{8}wl_1$	$\frac{1}{2}wl_2$	$\frac{1}{2}wl_2+\frac{H_3-H_2}{l_2}$	$\frac{1}{2}wl_2+\frac{H_3-H_2}{l_2}$
$A''_2+B''_2$	$\frac{3}{8}wl_1$	$\frac{1}{2}wl_2$	$\frac{1}{2}wl_2+\frac{H_2-H_3}{l_2}$	$\frac{1}{2}wl_2+\frac{H_2-H_3}{l_2}$
$A'_3+B'_3$	. .	$\frac{1}{2}wl_1-\frac{H_2}{l_1}$	$\frac{1}{2}wl_2+\frac{H_2-H_3}{l_2}$	$\frac{1}{2}wl_3$
$A''_3+B''_3$	. .	$\frac{1}{2}wl_1+\frac{H_2}{l_1}$	$\frac{1}{2}wl_2+\frac{H_3-H_2}{l_2}$	$\frac{1}{2}wl_3$
$A'_4+B'_4$	. .	. . .	$\frac{1}{2}wl_1-\frac{H_2}{l_1}$	$\frac{1}{2}wl_2+\frac{H_2-H_3}{l_2}$
$A''_4+B''_4$	. .	. . .	$\frac{1}{2}wl_1+\frac{H_2}{l_1}$	$\frac{1}{2}wl_2+\frac{H_3-H_2}{l_2}$
$A'_5+B'_5$	. .	. . .	. . .	$\frac{1}{2}wl_1-\frac{H_2}{l_1}$
$A''_5+B''_5$	. .	. . .	. . .	$\frac{1}{2}wl_1+\frac{H_2}{l_1}$

Stresses at supports for Symmetric Spans and Equable Loading.

Reactions	Two spans	Three spans	Four spans	Five spans
$A'_1+B'_1$	$\frac{3}{8}wl_1$	$\frac{1}{4}\frac{w}{l_1}\left(\frac{3l_1^3+6l_1^2l_2-l_2^3}{2l_1+3l_2}\right)$	$\frac{1}{4}\frac{w}{l_1}\left(\frac{6l_1^3+6l_1^2l_2-l_2^3}{4l_1+3l_2}\right)$	$\frac{1}{2}wl_1-\frac{1}{4}\frac{w}{l_1}\left\{\frac{(2l_2+3l_3)(l_1^3-l_2^3)-(l_2^3+l_3^3)}{(2l_2+3l_3)(2l_1+2l_2)-l_2^2}\right\}$
$A''_1+B''_1$	$\frac{5}{8}wl_1$	$\frac{1}{4}\frac{w}{l_1}\left(\frac{5l_1^3+6l_1^2l_2+l_2^3}{2l_1+3l_2}\right)$	$\frac{1}{4}\frac{w}{l_1}\left(\frac{10l_1^3+6l_1^2l_2+l_2^3}{4l_1+3l_2}\right)$	$\frac{1}{2}wl_1+\frac{1}{4}\frac{w}{l_1}\left\{\frac{(2l_2+3l_3)(l_1^3+l_2^3)-(l_2^3+l_3^3)}{(2l_2+3l_3)(2l_1+2l_2)-l_2^2}\right\}$
$A'_2+B'_2$	$\frac{5}{8}wl_1$	$\frac{1}{2}wl_2$	$\frac{1}{4}\frac{w}{l_1}\left(\frac{3l_1^3+6l_2^3+6l_1l_2^2}{4l_1+3l_2}\right)$	$\frac{1}{2}wl_2-\frac{3H_2(l_2+l_3)+\frac{1}{4}w(l_2^3+l_3^3)}{l_2(2l_2+3l_3)}$
$A''_2+B''_2$	$\frac{3}{8}wl_1$	$\frac{1}{2}wl_2$	$\frac{1}{4}\frac{w}{l_1}\left(\frac{-3l_1^3+6l_2^3+10l_1l_2^2}{4l_1+3l_2}\right)$	$\frac{1}{2}wl_2+\frac{3H_2(l_2+l_3)+\frac{1}{4}w(l_2^3+l_3^3)}{l_2(2l_2+3l_3)}$
$A'_3+B'_3$	. .	$\frac{1}{4}\frac{w}{l_1}\left(\frac{5l^3+6l_1^2l_2+l^3}{2l_1+3l_2}\right)$	$\frac{1}{4}\frac{w}{l_1}\left(\frac{-3l_1^3+6l_2^3+10l_1l_2^2}{4l_1+3l_2}\right)$	$\frac{1}{2}wl_3$
$A''_3+B''_3$	. .	$\frac{1}{4}\frac{w}{l_1}\left(\frac{3l_1^3+6l_1^2l_2-l_2^3}{2l_1+3l_2}\right)$	$\frac{1}{4}\frac{w}{l_1}\left(\frac{3l_1^3+6l_2^3+6l_1l_2^2}{4l_1+3l_2}\right)$	$\frac{1}{2}wl_3$
$A'_4+B'_4$	. .	. . . . .	$\frac{1}{4}\frac{w}{l_1}\left(\frac{10l_1^3+6l_1^2l_2+l_2^3}{4l_1+3l_2}\right)$	$\frac{1}{2}wl_2+\frac{3H_2(l_2+l_3)+\frac{1}{4}w(l_2^3+l_3^3)}{l_2(2l_2+3l_3)}$
$A''_4+B''_4$	. .	. . . . .	$\frac{1}{4}\frac{w}{l_1}\left(\frac{6l_1^3+6l_1^2l_2-l_2^3}{4l_1+3l_2}\right)$	$\frac{1}{2}wl_2-\frac{3H_2(l_2+l_3)+\frac{1}{4}w(l_2^3+l_3^3)}{l_2(2l_2+3l_3)}$
$A'_5+B'_5$	. .	. . . . .	. . . . .	$\frac{1}{2}wl_1+\frac{1}{4}\frac{w}{l_1}\left\{\frac{(2l_2+3l_3)(l_1^3+l_2^3)-(l_2^3+l_3^3)}{(2l_2+3l_3)(2l_1+2l_2)-l_2^2}\right\}$
$A''_5+B''_5$	. .	. . . . .	. . . . .	$\frac{1}{2}wl_1-\frac{1}{4}\frac{w}{l_1}\left\{\frac{(2l_2+3l_3)(l_1^3+l_2^3)-(l_2^3+l_3^3)}{(2l_2+3l_3)(2l_1+2l_2)-l_2^2}\right\}$

Also in the p^{th} span we have generally, when H_p is at p^{th} support,
$H_x = H_p + (A'_p + B'_p)x' - \frac{1}{2}wx'^2$; for any value of H and at any abscissa x'.

And H_m, the greatest value of H, is correspondingly

$$H_m = H_p + \frac{1}{2w}(A'_p + B'_p)^2 ;$$

the position of H_m in any span is given by the abscissa

$$x' = \frac{1}{w}(A'_p + B'_p); \text{ or by } x'' = \frac{1}{w}(A''_p + B''_p).$$

Also as $V'_p = A'_p + B'_p$; $V''_p = -(A''_p + B''_p)$ at the supports where they are greatest. In any span the p^{th}, we have $V_x = (A'_p + B'_p) - wx'$.

Thus the whole of the stresses necessary can be easily determined with the aid of the table ; although their values are not fully cleared in the instance of five spans, owing to the length of the expressions.

Deflexion.—Excepting in a few instances, it is absolutely necessary to have the values of all the spans, or at least their ratios given in numerical quantities, in order to arrive at convenient results. With a three-span girder an algebraic ratio, n, may be carried through the solution into the result.

A girder of two symmetrical spans is the case of two equal spans already treated.

A symmetric girder of three spans has merely the two outer spans equal ; or $l_1 = l_3$; for convenience denoting either of them by l simply, let $l_2 = nl$; where n is any fraction proper or improper.

In the first span we have, using the table,

$$H_x = 0 + \left(\tfrac{1}{2}wl + \frac{H_2}{l}\right)x - \tfrac{1}{2}wx^2$$

$$= \tfrac{1}{2}wlx - \tfrac{1}{4}wlx.\frac{n+1}{2+3n} - \tfrac{1}{2}wx^2$$

$$= \tfrac{1}{4}wlx.\frac{5n+3}{3n+2} - \tfrac{1}{2}wx^2 = EI.\delta^2{}_x y.$$

Equating this to zero, the abscissa x for the point of inflexion is $x = \frac{1}{2}l\left(\frac{5n+3}{3n+2}\right)$.

Now proceeding by integration,

$$\therefore\ EI.(\delta_x y - \tan\beta) = \frac{5n+3}{3n+2}.\tfrac{1}{8}wlx^2 - \tfrac{1}{6}wx^3 + C\,;$$

whence, as when $x = l$, $\delta_x y - \tan\beta = 0$; $C = -\frac{1}{24}wl^3.\frac{3n+1}{3n+2}$;

also from the equation for the second span. (See p. 177.)

$$-EI.\tan\beta = \tfrac{1}{6}nH_3 + \tfrac{1}{3}nlH_2 - \tfrac{1}{2}K_2$$

$$= -\tfrac{1}{2}nlH_2 + \tfrac{1}{24}wl^3 = -\tfrac{1}{8}wl^3x.\frac{n^2+n}{3n+2} + \tfrac{1}{24}wnl^3x\,;$$

$$\therefore\ EIy = \tfrac{1}{24}wlx^3.\frac{5n+3}{3n+2} - \tfrac{1}{24}wx^4 - \tfrac{1}{24}wl^3x.\frac{3n+1}{3n+2}$$

$$+ \tfrac{1}{8}wl^3x.\frac{n^2+n}{3n+2} - \tfrac{1}{24}wl^3xn.$$

or $EIy = \frac{1}{24}wlx^3.\frac{5n+3}{3n+2} - \frac{1}{24}wx^4 + \frac{1}{24}wl^3x.\frac{8n+1}{3n+2}$;

the equation to the curve of the neutral axis, which may be plotted, to obtain the flexure ξ.

Or, if a numerical value be given to n, and introduced in the value of $\delta_x y = 0$, the abscissa x corresponding to ξ may be obtained; then substituting this x in the equation to the curve, the value of $y = \xi$ is analytically deduced.

In the middle span we have, using the table,

$$H_x = H_2 + \tfrac{1}{2}wl_2x - \tfrac{1}{2}wx^2$$
$$= -\tfrac{1}{4}wl^2.\frac{n+1}{3n+2} + \tfrac{1}{2}wnlx - \tfrac{1}{2}wx^2 = EI.\delta_x^2y.$$

From equating this to zero, the abscissa x for the points of inflexion is

$$x = \tfrac{1}{2}l\left\{n \pm \left(\frac{3n^3+2n^2-2n-2}{3n+2}\right)^{\frac{1}{2}}\right\};$$

$$\therefore\ EI.(y+\tan\beta) = -\tfrac{1}{4}wl^2x.\frac{n+1}{3n+2} + \tfrac{1}{4}nwlx^2 - \tfrac{1}{6}wx^3 + C;$$

from the condition $y+\tan\beta=0$, when $x=l$,

$$C = \tfrac{1}{24}wl^3\left\{\frac{-18n^2+6n+14}{3n+2}\right\};$$

also as before, $+\tan\beta = +\tfrac{1}{8}wl^3x\dfrac{n^2+n}{3n+2} - \tfrac{1}{24}wl^3x.n$;

$$\therefore\ EI.y = -\tfrac{1}{8}wl^2x^2.\frac{n+1}{3n+2} + wlx^3.\frac{n}{12} - \tfrac{1}{24}wx^4 \ \ldots$$
$$+\tfrac{1}{24}wl^3x\left\{\frac{-18n^2+6n+14}{3n+2}\right\} - \tfrac{1}{8}wl^3x\frac{n^2+n}{3n+2} + \tfrac{1}{24}wl^3x.$$

$$\therefore\ EIy = -\tfrac{1}{8}wl^2x^2.\frac{n+1}{3n+2} + wlx^3.\frac{n}{12} = \tfrac{1}{24}wx^4 + \tfrac{1}{24}wl^3x\left\{\frac{-21n^2+6n+16}{3n+2}\right\}$$

the equation to the curve of the neutral axis.

From symmetry, the abscissa x for the flexure ξ will be $\tfrac{1}{2}nl$ or at midspan, hence

$$\xi = \frac{wl^4}{32EI}\left\{\frac{-45n^3+9n^2+32n}{3(3n+2)} + \frac{n^4}{4}\right\}.$$

The Maximum Stresses.—In the first span,

$$H_m = H_1 + \frac{1}{2w}\left\{\frac{wl^3}{4l^2}\left(\frac{3+6n-n^3}{3n+2}\right)\right\}^2 = \tfrac{1}{2}l^2w.\left\{\frac{3+6n-n^3}{3n+2}\right\}^2;$$

the position of H_m is given by the abscissa or value of x,

$$x=\frac{1}{w}\left\{\frac{wl^3}{4l^2}\left(\frac{3+6n-n^3}{3n+2}\right)\right\}=\tfrac{1}{4}l\left\{\frac{3+6n-n^3}{3n+2}\right\};$$

also $V_m=\tfrac{1}{4}wl.\left(\frac{3+6n-n^3}{3n+2}\right)$,

its position being at the support.

The negative maxima of H_m and V_m are of less consequence.

In the second span,

$$H_m=H_2+\frac{1}{2w}\{\tfrac{1}{2}wln\}^2=-\tfrac{1}{4}wl^2.\frac{n+1}{3n+2}+\tfrac{1}{8}wl^2n^2$$
$$=\tfrac{1}{8}wl^2\left\{n^2-\frac{2n+2}{3n+2}\right\};$$

the position of H_m is given by the abscissa or value of x,

$$x=\frac{1}{w}(\tfrac{1}{2}wln)=\tfrac{1}{2}nl;$$

also $V_m=\tfrac{1}{2}wnl$, its position being at the support.

The same method may be applied to girders of spans up to five in number with the help of the Table, if numerical ratios or values be used for the varied lengths of span, l_1, l_2, l_3.

GENERAL SOLUTION.

Solution Number 23.—Continuous girder of any number of equal spans, under equally distributed several loadings, the intensity of load being symmetrically disposed with reference to the middle of the whole.

Here $w_1=w_n$; $w_2=w_{n-1}$; $w_3=w_{n-2}$, *&c.*; and the mode of solution will be analogous to that before adopted.

By the general theorem reduced for these conditions,

$$H_1l_1+2H_2(l_1+l_2)+H_3l_2=3(K_1+K_2)=-\tfrac{1}{4}w_1l_1^3-\tfrac{1}{4}w_2l_2^3.$$
$$H_1+4H_2+H_3=-\tfrac{1}{4}l^2(w_1+w_2).$$

Applying this to any number (n) of equal spans, l,

$$
\begin{aligned}
&H_1=H_{n+1} &&=0 \text{ at the free ends,} \\
&4H_2+H_3 &&=H_{n-1}+4H_n &&=-\tfrac{1}{4}l^2(w_1+w_2) \\
&H_2+4H_3+H_4 &&=H_{n-2}+4H_{n-1}+H_n &&=-\tfrac{1}{4}l^2(w_2+w_3) \\
&H_3+4H_4+H_5 &&=H_{n-3}+4H_{n-2}+H_{n-1} &&=-\tfrac{1}{4}l^2(w_3+w_4) \\
& &&\&c.=\&c.
\end{aligned}
$$

The condition of symmetry also gives

$$H_2=H_n;\ H_3=H_{n-1};\ H_4=H_{n-2};H_5=H_{n-3},\ \&c.=\&c.$$

Hence the values of all these horizontal stress moments at the supports may be obtained. See Table following.

The reactions $A'+B'$ and $A''+B''$ to the right and left of each span may now be obtained in the mode of Solution Number 21 (p. 182) and tabulated.

Using the table, we have generally in the p^{th} span, when H_p is at the p^{th} support and x is any abscissa, $H_x=H_p+(A'_p+B'_p)x'-\frac{1}{2}wx'^2$, for any value of H; and H_m the greatest value of H_x is correspondingly

$$H_m=H_p+\frac{1}{2w}(A'_p+B'_p)^2.$$

The position of H_m in any span is given by the abscissa

$$x'=\frac{1}{w}(A'_p+B'_p);\ \text{or by that from the left}\ x''=\frac{1}{w}(A''_p+B''_p).$$

Also as $V'_p=A'_p+B'_p$; $V''_p=-(A''_p+B''_p)$ at the two supports where the values are greatest, these are therefore corresponding values of V_m.

In any span the p^{th} we have at any abscissa x,

$$V_x=(A'_p+B'_p)-wx'.$$

In the above formulæ w is the load intensity belonging to the span of reference, whichever it may be; and may be written w_p when necessary for the sake of distinction.

Stresses at Supports for Symmetric Loading and Equal Spans.

Horizontal Stress	Two Spans	Three Spans	Four Spans	Five Spans
H_1	o	o	o	o
H_2	$-\frac{1}{16}l^2(w_1+w_2)$	$-\frac{1}{20}l^2(w_1+w_2)$	$-\frac{1}{14}l^2(w_1+\frac{3}{4}w_2-\frac{1}{4}w_3)$	$-\frac{1}{76}l^2(5w_1+4w_2-w_3)$
H_3	o	$-\frac{1}{20}l^2(w_1+w_2)$	$-\frac{1}{28}l^2(-w_1+w_2+2w_3)$	$-\frac{1}{76}l^2(-w_1+3w_2+4w_3)$
H_4	— —	o	$-\frac{1}{14}l^2(w_1+\frac{3}{4}w_2-\frac{1}{4}w_3)$	$-\frac{1}{76}l^2(-w_1+3w_2+4w_3)$
H_5	— —	— — —	o	$-\frac{1}{76}l^2(5w_1+4w_2-w_3)$
H_6	— —	— — —	— — — —	o
Reactions				
$A'_1+B'_1$	$\frac{1}{16}l(7w_1-w_2)$	$\frac{1}{20}l(9w_1-w_2)$	$\frac{1}{14}l(6w_1-\frac{3}{4}w_2+\frac{1}{4}w_3)$	$\frac{1}{76}l(33w_1-4w_2+w^3)$
$A''_1+B''_1$	$\frac{1}{16}l(9w_1+w_2)$	$\frac{1}{20}l(11w_1+w_2)$	$\frac{1}{14}l(8w_1+\frac{3}{4}w_2-\frac{1}{4}w_3)$	$\frac{1}{76}l(43w_1+4w_2-w_3)$
$A'_2+B'_2$	$\frac{1}{16}l(9w_1+w_2)$	$\frac{1}{2}lw_2$	$\frac{1}{56}l(+6w_1+29w_2-5w_3)$	$\frac{1}{76}l(+6w_1+39w_2-5w_3)$
$A''_2+B''_2$	$\frac{1}{16}l(7w_1-w_2)$	$\frac{1}{2}lw_2$	$\frac{1}{56}l(-6w_1+27w_2+5w_3)$	$\frac{1}{76}l(-6w_1+37w_2+5w_3)$
$A'_3+B'_3$	— —	$\frac{1}{20}l(11w_1+w_2)$	$\frac{1}{56}l(-6w_1+27w_2+5w_3)$	$\frac{1}{2}lw_3$
$A''_3+B''_3$	— —	$\frac{1}{20}l(9w_1-w_2)$	$\frac{1}{56}l(+6w_1+29w_2-5w_3)$	$\frac{1}{2}lw_3$
$A'_4+B'_4$	— —	— — —	$\frac{1}{14}l(8w_1+\frac{3}{4}w_2-\frac{1}{4}w_3)$	$\frac{1}{76}l(-6w_1+37w_2+5w_3)$
$A''_4+B''_4$	— —	— — —	$\frac{1}{14}l(6w_1-\frac{3}{4}w_2+\frac{1}{4}w_3)$	$\frac{1}{76}l(+6w_1+39w_2-5w_3)$
$A'_5+B'_5$	— —	— — —	— — — —	$\frac{1}{76}l(43w_1+4w_2-w_3)$
$A''_5+B''_5$	— —	— — —	— — — —	$\frac{1}{76}l(33w_1-4w_2+w_3)$

Deflexions.—A girder of three equal spans, with the load equally distributed on each span, and the load-intensities symmetrical on each side of the middle of the girder.

Here $w_1 = w_3 = w$ suppose; and $w_2 = mw$, where m is any given fraction proper or improper.

In the first span we have, using the Table

$$H_x = 0 + \tfrac{1}{20}l.x(9w_1 - w_2) - \tfrac{1}{2}w_1x^2 = \tfrac{1}{20}wlx(9-m) - \tfrac{1}{2}wx^2.$$

Equating this to zero, the abscissa of the point of inflexion is $x = \frac{1}{10}l.(9-m)$.

Equating $H_x = EI.\partial_x^2 y$, and integrating between 0 and l,

$EI.(\partial_x y - \tan\beta) = \frac{1}{40}wlx^2(9-m) - \frac{1}{6}wx^3 + C,$

and from the condition that $\partial_x y - \tan\beta = 0$, when $x = l$

$$C = -\tfrac{1}{120}(7-3m)wl^3.$$

Also from the second span,

$$EI.\tan\beta = \tfrac{1}{6}nlH_3 + \tfrac{1}{3}nlH_2 - \tfrac{1}{2}K_2$$
$$= -\tfrac{1}{40}l^3w(1+m) + \tfrac{1}{24}wml^3 = -\tfrac{1}{120}wl^3.(3-2m)$$
$$\therefore\ EIy = \tfrac{1}{120}wlx^3(9-m) - \tfrac{1}{24}wx^4 - \tfrac{1}{120}wl^3x(+4-m)$$

the equation to the curve of the neutral axis.

This may be plotted to obtain the flexure ξ or greatest value y, and its abscissa. Or, if a numerical value be assigned to m, this abscissa may be obtained from the equation $\partial_x y = 0$; and applying that abscissa x in the equation to the curve, ξ can be found.

In the middle span, we have by using the Table

$$H_x = H_2 + \tfrac{1}{2}lmnx - \tfrac{1}{2}mwx^2$$
$$= -\tfrac{1}{20}l^2w(1+m) + \tfrac{1}{2}lw.mx - \tfrac{1}{2}mwx^2 = EI\partial_x^2 y.$$

Equating this to zero, the abscissa for the point of inflexion is found

$$x = \frac{l}{2\sqrt{5}}\left\{5m \pm (2m+2)\right\}^{\frac{1}{2}};$$

and integrating between x and o

$EI(\partial_x y - \tan\beta) = -\frac{1}{20}wl^2x(1+m) + \frac{1}{4}wmlx^2 - \frac{1}{6}mwx^3 + C,$

where from the condition $\partial_x y - \tan\beta = 0$, when $x = l$,

$C = +\frac{1}{20}wl^3 - \frac{1}{30}mwl^3$;

and as before obtained, $\tan\beta = -\frac{1}{120}w\,l^3(3-2m)$;

$\therefore\ EIy = -\frac{1}{40}wl^2x^2(1+m) + \frac{1}{12}wmlx^3 - \frac{1}{24}mwx^4 + \frac{1}{20}wl^3x$
$\qquad - \frac{1}{30}mwl^3x + \frac{1}{120}wl^3x(3-2m)$;

$\therefore\ EIy = -\frac{1}{40}wl^2x^2(1+m) + \frac{1}{12}wlx^3m - \frac{1}{24}wx^4m + \frac{1}{120}wl^3x(3-2m),$

the equation to the curve of the neutral axis.

From symmetry of loading, the abscissa x for flexure ξ will be $\frac{1}{2}ml$ or at midspan, hence

$$\xi = \frac{wl^4}{1920EI}\cdot\left\{24m - 38m^2 - 12m^3 + 20m^4 - 5m^5\right\}.$$

The Maximum Stresses.—In the first span

$$H_m = H_1 + \frac{1}{2w}\left\{\frac{1}{20}lw(9-m)\right\}^2 = \frac{wl^2}{800}(9-m)^2;$$

the position of H_m is given by the abscissa or value of x,

$$x = \frac{1}{w}\left\{\frac{1}{20}lw.(9-m)\right\} = \frac{1}{20}l(9-m);$$

also $V_m = \frac{1}{20}lw(9-m)$;

its position being at the point of support.

The negative maxima for H_m and V_m are of less consequence.

In the second span

$$H_m = H_2 + \frac{1}{2w}\left\{\frac{1}{2}lmw\right\}^2 = \frac{wl^2}{40}(5m^2 - 2m - 2);$$

the position of H_m is given by the abscissa or value of x,

$$x = \frac{1}{w}(\tfrac{1}{2}lmw) = \tfrac{1}{2}lm.$$

Also $V'_m = \frac{1}{2}lmw$, its position being at the support.

The same method may be applied to girders of spans up to five in number with the help of the Table, provided that numerical ratios or values be used for the various load-intensities, w_1, w_2, w_3.

Solution Number 24.—*Special cases of continuous girders of unequal spans and unsymmetrical loading.*

Any general solution for all cases up to five or six spans would be lengthy and complicated. Cases of two or three spans, being alone commonly required, will be here treated singly; the same method can, however, be applied to any number of spans and variety of load-intensity, by reverting to the General Theorem.

Two-span girder.—With a two-span continuous girder having spans l_1 and l_2, and load-intensities w_1 and w_2.

$$\text{As } H_1 = H_3 = 0 \,; \quad H_2 = -\tfrac{1}{8} . \frac{w_1 l_1^3 + w_2 l_2^3}{l_1 + l_2} \,;$$

$$A'_1 + B'_1 = \tfrac{1}{2} w_1 l_1 + \frac{H_2}{l_1} \,; \text{ and } A''_1 + B''_1 = \tfrac{1}{2} w_1 l_1 - \frac{H_2}{l_1} \,;$$

$$A'_2 + B'_2 = \tfrac{1}{2} w_2 l_2 = \frac{H_2}{l_2} \,; \text{ and } A''_2 + B''_2 = \tfrac{1}{2} w_2 l_2 + \frac{H_2}{l_2} .$$

$$\text{In the first span } H_x = \left(\tfrac{1}{2} w_1 l_1 + \frac{H_2}{l_1} \right) x - \tfrac{1}{2} w_1 x^2 \,;$$

$$\text{in the second span } H_x = \left(\tfrac{1}{2} w_2 l_2 - \frac{H_2}{l_2} \right) x - \tfrac{1}{2} w_2 x^2 .$$

$$\text{In the first span } H_m = \frac{1}{2 w_1} \left\{ \tfrac{1}{2} w_1 l_1 + \frac{H_2}{l_1} \right\}^2 \,;$$

$$\text{in the second span } H_m = \frac{1}{2 w_2} \left\{ \tfrac{1}{2} w_2 l_2 + \frac{H_2}{l_2} \right\}^2 .$$

$$\text{In the first span } V_x = \tfrac{1}{2} w_1 l_1 + \frac{H_2}{l_1} - w_1 x \,;$$

in the second span $V_x = \frac{1}{2} w_2 l_2 - \frac{H^2}{l_2} w_2 x.$

At H_1, $V_m = A'_1 + B'_1$: at H_2, $V_m = -(A''_1 + B''_1)$:
at H_3, $V_m = -(A''_2 + B''_2)$; also at H_2, $V_m = +(A'_2 + B'_2)$.

The points of inflexion are given by the abscissæ

in the first span, $x = \frac{2}{w_1}(A'_1 + B'_1)$;

in the second span, $x = \frac{2}{w_2}(A''_2 + B''_2)$.

With these values, and putting $l_2 = n l_1$; $w_2 = m w_1$; and having numerical values of either m or n, the equation to the curve of the neutral axis and the flexures may be reduced as before in the preceding cases.

Three-span girder.—With unequal loading and symmetric spans,

if $l_1 = l_3$, but w_1, w_2, w_3, are all different, we may obtain

$$H_2 = -\frac{2w_1 l_1^3 (l_1 + l_2) + w_2 l_2^3 (2l_1 + l_2) - w_3 l_3^3 . l_2}{4(2l_1 + 3l_2)(2l_1 + l_2)}; \qquad H_1 = 0;$$

$$H_3 = -\frac{2w_3 l_1^3 (l_1 + l_2) + w_2 l_2^3 (2l_1 + l_2) - w_1 l_1^3 . l_2}{4(2l_1 + 3l_2)(2l_1 + l_2)}; \qquad H_4 = 0;$$

and proceed in the same manner as in the last case. As the results in these cases become complicated and correspondingly liable to error, it is more usual to adopt either equal spans or equal load-intensities; they can then be rapidly solved according to the mode in Solutions Numbers 22 and 23 with the help of the tabulated stresses.

But if l_1 and l_2 are different, the most useful application of this case will occur in a swing-bridge.

Pivoted Swing-bridge.—This may be a case of unequal continuous spans with unequal loads; for when the bridge is closed, the ends are supported; there are then three spans, of which the middle span is comparatively unloaded, and four points of support.

Applying the General Theorem and putting $H_1 = H_4 = 0$; and $w_2 = 0$; we have

$$2H_2(l_1 + l_2) + H_3 l_2 = -\tfrac{1}{4} w_1 l_1^3; \quad H_2 l_2 + 2H_3(l_2 + l_3) = -\tfrac{1}{4} w_3 l_3^3;$$

whence $H_2 = \dfrac{-2w_1 l_1^3(l_2 + l_3) + w_3 l_3^3 l_2}{4l_2(4l_1 + 3l_2) + 16l_3(l_1 + l_2)}$;

and $H_3 = \dfrac{w_1 l_1^3 l_2 - 2w_3 l_3^3(l_1 + l_2)}{4l_2(4l_1 + 3l_2) + 16l_3(l_1 + l_2)}$;

also $(A + B)_1 = \tfrac{1}{2} w_1 l_1 + \dfrac{H_2}{l_1}$; $(A + B)_4 = \tfrac{1}{2} w_3 l_3 + \dfrac{H_3}{l_3}$;

values that may be applied in the manner before exemplified for finding H_x and V_x at any abscissa x in any span.

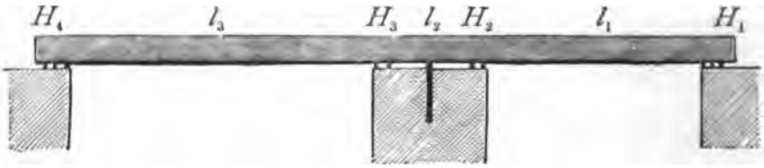


FIGURE 19.

Also if $l_2 = 0$, and $H_2 = H_3$, by altering the subscripts the above may after reduction be applied to the resulting case of a two-span continuous girder with unequal spans and unequal uniform loadings, as before explained.

Solution Number 25.—*The effect of passing load on a continuous girder of two and of three spans.*

This subject has been ably treated by Cunningham in his 'Applied Mechanics,' pages 372 to 377. The following is principally due to him.

Two-span girder.—With a girder of two unequal spans l_1, l_2, the load-intensity w being the same when passing each span, and the ordinary load being neglected.

The chief object is to find the extreme possible values

of shearing force and bending moment denoted by V_{cm} and H_{cm} that can occur during the passage of the load. Before proceeding to these it will be necessary to notice the variations in K_1 and K_2, those in H_2, and those in $(A'+B')_1$, $(A''+B'')_1$, $(A'+B')_2$, $(A''+B'')_2$.

In the loaded segment c_1, next to the outer support,

$$K=\frac{wc_1^2}{12l}(c_1^2-2l^2)=-\frac{w}{12l}\left\{l^4-(l^2-c_1^2)^2\right\}.$$

In the loaded segment c_2, next to the middle support,

$$K=-\frac{w}{12l}(l^2-c_1^2)^2=-\frac{w}{12l}\left\{l^2-(l-c_2^2)\right\}^2;$$

where l is the span of reference, either l_1 or l_2.

Hence $-K$ increases with the length of the loaded segment and is greatest when $c_1=l$, or $c_2=l$, that is where the span is fully loaded, and then $K=-\frac{1}{12}wl^3$.

As to the effect on H_2.

Since $2H_2(l_1+l_2)=3(K_1+K_2)$, it is evident that $-H_2$ increases with the extent of load, and is a maximum when l_1 and l_2 are both fully loaded.

As to the effect on the reactions at the supports.

It is evident that A'_1, A''_1, A'_2, A''_2 increase with the load on l_1 and l_2 respectively, and are positive;

also as $B''_1=-\frac{H_2}{l_1}$, and $B'_2=-\frac{H_2}{l_2}$, of which $-H_2$ is always positive and increases with the load; hence $(A''+B'')_1$ and $(A'+B')_2$ are always positive and increase with the load.

Again, as $B'_1=+\frac{H_2}{l_1}$, and $B''_2=+\frac{H_2}{l_2}$, and H_2 is naturally negative, hence $(A'+B')_1$ and $(A''+B'')_2$ may be either positive or negative. It will suffice to trace the variation of the former of them.

Since $(A+B)_1 = A_1 + \frac{3(K_1+K_2)}{2l_1(l_1+l_2)}$, and the two cases of load on c_1 and on c_2 require separate treatment,

(1) $$(A'+B')_1 = \frac{wc_1(2l_1-c_1)}{2l_1} - \frac{wc_1^2(2l_1^2-c_1^2)}{8l_1^2(l_1+l_2)} + \frac{3K_2}{2l_1(l_1+l_2)}$$
$$= \frac{w}{2l_1}\{l_1^2-(l_1-c_1)^2\} - w\frac{l_1^4-(l_1^2-c_1^2)^2}{8l_1^2(l_1+l_2)} + \frac{3K_2}{2l_1(l_1+l_2)};$$

where the two first terms are together positive, increasing with $c_1 (c_1 < l_1)$, and the last is negative.

(2) $$A'_1+B'_1 = \frac{wc_2^2}{2} - \frac{wc_2^2(2l_1-c_2)^2}{8l_1^2(l_1+l_2)} + \frac{3K_2}{2l_1(l_1+l_2)}$$
$$= \frac{wc_2^2}{2l_1}\left\{1 - \frac{(2l_1-c_2)^2}{4l_1(l_1+l_2)}\right\} + \frac{3K_2}{2l_1(l_1+l_2)};$$

of which the first term is positive, increasing with c_2, and the last is negative.

Combining these results, it follows that $A'_1+B'_1$ is a negative maximum when l_1 is unloaded and l_2 is fully loaded, and that it is a positive maximum when l_2 is unloaded and l_1 is fully loaded. Corresponding results will also hold with $A''_2+B''_2$.

Proceeding now to the variation of V.

On the span l_1, V_{cm} will exist when its longer segment is loaded.

(1) Its greatest positive value, near H_1, when the longer segment c_2, measured from the middle support, is fully loaded, and when l_2 is unloaded, is

$$V_{cm} = A'_1 + B'_1 = \frac{wc_2^2}{2l_1}\left\{1 - \frac{(2l_1-c_2)^2}{4l_1(l_1+l_2)}\right\}$$

(2) Its greatest negative value, near H_2, when the longer segment c_1, measured from the first support, is fully loaded, and when l_2 is fully loaded, is

$$V_{em} = -(A'' + B'')_1 = -\left\{A''_1 - \tfrac{3}{2}.\frac{(K_1+K_2)}{l_1(l_1+l_2)}\right\}$$
$$= -\left\{w.\frac{c_1^2}{2l_1} + \frac{wc_1^2(2l_1^2 - c_1^2)}{8l_1^2(l_1+l_2)} + \frac{wl_1^2}{8(l_1+l_2)}\right\}.$$

Corresponding results will also hold on the span l_2 if the sign of V_{em} be suitably changed.

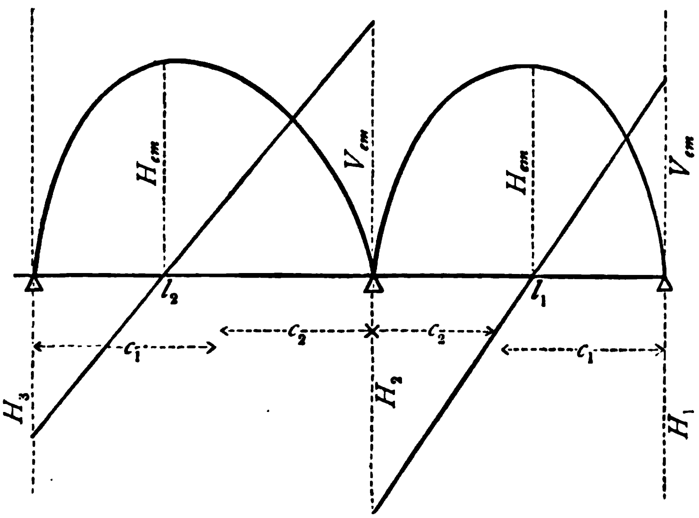


FIGURE 20.—Extreme Stress under Passing Load.

Proceeding to the variation of H.

On the span l_1, H_{em} will exist at every section, thus:

(1) Its greatest positive value, near H_1, when l_1 is loaded, and l_2 is unloaded.

$H_{em} = (A'_1 + B'_1)c_1 - \frac{1}{2}wc_1^2.$

(2) Its greatest negative value, near H_1, when l_2 is loaded, and l_1 is unloaded.

$H_{em} = (A'_1 + B'_1)c_1.$

(3) Its greatest negative value, near H_2, when both spans are fully loaded.

$H_{cm}=\frac{c_1}{l}H_2+H_d$; where H_d is the moment due to the span treated as discontinuous.

Corresponding results will also hold on the span l_2.

Three-span girder.—With a girder of three spans of lengths symmetric, $l_1=l_3$, and a passing load-intensity w, neglecting the permanent load.

Following the method adopted in the last, the effect of the passing load on K_1, K_2 will be greatest when the span is fully loaded, or when K_1 or K_2 is a maximum, $-\frac{1}{12}wl_1^3$, or $-\frac{1}{12}wl_2^3$.

Next, the effect on H_2 and H_3.

From a consideration of the tabular values of H_2 and H_3 (see page 190) it is seen that $-H_2$ and $-H_3$ are maxima when the nearest outer and the middle span is fully loaded, and the farthest outer span is unloaded; also that $+H_2$, $+H_3$ are maxima respectively when their two adjoining spans respectively are unloaded and the furthest outer span is loaded.

As to the effect on the reactions at the supports.

$$A'_1+B'_1=\tfrac{1}{2}wl+\frac{H_2}{l}\,;\ \text{and}\ A''_3+B''_3=\tfrac{1}{2}wl+\frac{H_3}{l}\,;$$

these are maxima when l_2 is unloaded, and l_1 and l_3 are fully loaded.

$A''_1+B''_1$ is a maximum when l_3 is unloaded, and l_1 and l_2 are fully loaded; and $A'_3+B'_3$ is a maximum when l_1 is unloaded, and l_2 and l_3 are fully loaded.

$A'_2+B'_2$ is a maximum when l_3 is unloaded, and $A''_2+B''_2$ when l_1 is unloaded; the remaining spans being fully loaded.

Proceeding now to the variation of V.

The extreme possible value of V, denoted by $\pm V_{ex}$,

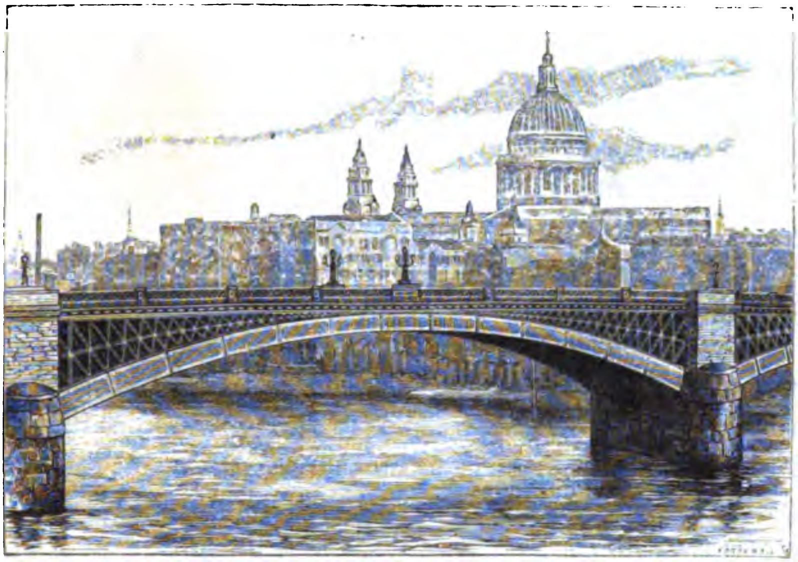
will attain in any span where one or other of the segments extending up to the section under consideration is fully loaded and the other unloaded, and all the remaining spans are so loaded as to cause the reaction at the end of the unloaded segment to be a maximum.

Proceeding now to the variation of H.

H will attain the value H_{em} at every section in each span, under the following conditions:—

Span		Conditions for $+H_{em}$	Conditions for $-H_{em}$
1st.	In l_1 near abutment	l_1 and l_3 loaded, l_2 unloaded	l_2 and l_3 loaded, l_1 unloaded
	In l_1 near pier . .	None	l_1 and l_2 loaded, l_3 unloaded
2nd.	In l_2 not near piers .	l_2 loaded, l_1 and l_3 unloaded	l_1 and l_3 loaded, l_2 unloaded
3rd.	In l_3 near pier . .	None	l_2 and l_3 loaded, l_1 unloaded
	In l_3 near abutment	l_1 and l_3 loaded, l_2 unloaded	l_1 and l_2 loaded, l_3 unloaded

In the figures attached to the foregoing solutions the representative positions of extreme stresses have been added in accordance with Cunningham's method.



SECTION II.

CURVED RIBS; AND METALLIC ARCHES.

The Curved Rib. General Conditions.

In Part I., when treating of the static resolution of stress, the development of the Curved Rib from the bow-string girder was explained; and perhaps that followed the more rational mode; the more natural one, and that more in accordance with fact, would be to develop it from the rigid or masonry arch. In this latter process it would take two stages: first, from the masonry arch to the metallic arch, which consists of framed or bolted voussoirs of metal; next, from the pieced metallic arch to the continuous curved rib; the former being most frequently of cast iron, the latter necessarily of wrought iron or steel.

If we conceive an arch, or a narrow strip of arch, to be transformed from a collection of pieces, whether cemented

or bolted blocks, to a perfectly homogeneous whole; or if we imagine the whole to consist of parts so connected that the joints are exactly similar to the mass as regards resistance of every sort, we have then a continuous whole, or curved rib. This is evidently no longer an arch, for it is unaffected by the principles holding with block-structures, and the curved form alone remains.

But, following the physical transformation from one to the other, it will be noticed that in a true arch the line of resistance or of thrust representing the whole series of resistances or thrusts is a broken curve drawn through points at each joint (see fig. p. 42), and the transmission of stress through each block is dispersed, though theoretically treated as rectilinear, between the aforesaid points. In the curved rib, on the contrary, the line of resistance is a true unbroken curve, and is there correctly termed a curve of equilibrium, under conditions of equilibrium.

The correspondence in the two cases is true, but the distinction is equally well marked.

Curved ribs of this general simple type comprise solid and nearly rigid concrete ribs, also elastic laminated ribs either of timber or of riveted plate iron, so disposed as to afford effective continuity, with rigid connections not partaking of the nature of radiated or pseudo-radiated joints. Also in some such cases, nominal upper members are attached to the curved rib as intermediaries in sustaining load or roadway, transmitting direct weight, but not sustaining stress in other respects, nor sharing the work of the curved rib. Such upper portions are not true members of it, but are merely extraneous parts.

In other cases, curved ribs have true upper members that sustain both burden and thrust in unison with the main curved member. Beyond, there may also exist some

spandrel-filling or rigid spandrel-bars between the upper and the main member: these are termed compound curved ribs, and may be open, pierced, or rigidly connected. If, however, the connections consist of articulated or free bracing, the structure is then a Braced Girder, and belongs to a separate type.

Simple curved ribs, though continuous in every respect, may either have uniform section, or may vary in depth so much that though the intrados is curved the extrados may be horizontal, or may take any form intermediate between the two.

The treatment of the curved rib depends on the consideration both of its curve of equilibrium and of its elastic deformation under stress, thus partaking of the treatment of a beam or girder. The neutral curve, roughly imagined to pass through the centre of gravity of its cross sections everywhere, is inconstant, even when it is a curve of equilibrium; hence the difficulties attending its analytical determination.

When the curved rib is perfectly continuous and not hinged or pivoted at the crown, the position of the pressure-curve at the crown being then unknown, the rise of that curve is also unknown. It then is impossible to apply the elementary method of static resolution to the determination of abutment-reactions and unknown quantities with precision generally. Such a mode, which is usually adopted with braced girders having articulated bracing, can only be used approximately with curved ribs of very nearly uniform depth, and of comparatively small depth, and with the aid of much discriminative power. With elastic curved ribs of variable depth it is necessary to employ the equations of elastic deformation for the determination of unknown reactions and other quantities.

The former method is dependent on (what is termed) the effective depth of the rib being known, when the rib has an upper and a lower flange-member, or is composite in structure. It is applied to ribs of circular curvature in a set of solutions following. The latter method is a general method dependent on an elaborated General Theory, which will also be given afterwards, applying to all curved ribs of any curvature.

The Curvature.—The curve of equilibrium in a curved rib is necessarily dependent on the distribution of the load, and on the ratio of the load to the inherent weight of the rib.

If this ratio be constant, and within ordinary limits, and if the loading rise to the level of a horizontal platform or roadway, the general condition is roughly thus :—

	Curve
With a load applied vertically	Parabola
With load applied radially	Circular
With radial load and filled haunches . . .	Elliptic

There is also some analogy between the curvature of a curved rib and that of suspension chains under corresponding conditions, thus :

With constant section and load proportioned to length	Similar curvature
With constant section and comparatively trifling load	Common catenary
With section varying with the load .	Catenary of equal strength

Thus, with analogous load and section, the curvature of a suspension chain and of a curved rib may be similar in these cases.

Yet with various ratios of load to inherent weight the

curvature of a curved rib may take any form intermediate between any of those above mentioned.

It is evident that in bridges, roofs, &c., where any of the above conditions of loading may exist, the curvature most suited to the rib itself is that generally harmonising with the anticipated curve of equilibrium ; yet constructive convenience most frequently induces the adoption of circular curvature.

The same reason applies under the same conditions to the employment of uniform or approximately uniform depth of section, varied merely by the thickness of two or three additional plates, in preference to sections of great and continual variation; the converse being exceptional rather than usual. For, on examining the calculated extreme stresses on large curved ribs of circular curvature under partial as well as uniformly distributed load, even with allowance for temperature strains, it may be noticed that such extreme stresses generally vary but slightly along two-thirds of each span ; that they increase near the abutments to a considerable degree for about a quarter of each half-span, and that they diminish at the crown throughout the remaining portion.

Such variation practically demands only three varieties of section throughout the rib of circular curvature, or even only two, if it be preferred not to lighten the crown ; and the variation in section may often be satisfied by merely some plate-thicknesses. This actual development is a practical intermediate between strictly uniform depth and continuously varying sections that holds sufficiently well under dimensions, loads, and conditions, that are large and extreme compared with those usually accompanying ordinary bridge-ribs and roof-ribs.

As to choice of section, it may be noticed that a con-

dition favourable to a section is that its radius of gyration shall be comparatively large while its depth is relatively small. (The abstract relations between d the depth and $r=\left(\frac{I}{S}\right)^{\frac{1}{2}}$ for various sections are too varied to be of much use, so numerical reduction for each section is necessary.) But this, based on theoretic conclusions, is wholly independent of constructive convenience.

The rise of a curved rib, or rather the ratio $\frac{h}{l}$ of rise to span, is necessarily much dependent on local circumstances and conditions; but with a bridge-rib the ratio $\frac{1}{10}$ is that most usual. M. Bresse, in his 'Mécanique Appliqué,' is said to enter into theoretic considerations for basing the ratio most favourably in relation to stress; but any arguments for or against a higher or a lower ratio within moderate limits can hardly succeed in defining any exact ratio, that would not usually be set aside on account of any local or particular convenience.

These foregoing deductions combine to show that the form and conditions of curved ribs most commonly required are those adopted in the following Solutions Nos. 1, 2, 3, 4. In other cases the General Theory, No. 5, following them, can be specially applied, in the mode of No. 6.

Distinctive Terms.—As there remains still much confusion of terms applied to curved ribs, braced girders, and iron or metallic arches, and as the composite structures themselves in some cases partake of the natures of more than one of these distinctive types, it may be convenient to express the general signification of the terms, and the principles on which the distinctions are made.

(1) A girder is necessarily continuous in its upper and lower flange-members, its web may be either continuous or

braced, the bracing having free articulation. In the latter case it is a braced girder, and it may be either horizontal with uniform depth or of unequal depth with curved members.

(2) A curved rib has necessarily a continuous curved member ; but when compound, it may have spandrel-bars or spandrel-filling between its upper and its lower curved member, the spandrel-bars being fixed and not articulated.

(3) A metallic arch necessarily consists of framed panels (corresponding to the voussoirs of a masonry arch), which are bolted or dowelled together at the bearings or joints ; there are hence not any strictly continuous upper and lower members in a simple metallic arch.

When these distinctions actually and clearly exist, it is unpardonable to confound the terms applied to such structures ; the coarse mistake of terming a structure an arch, when it has no representative voussoirs whatever, is a habit now becoming obsolete.

Solution Number 1.—*Fixed rib of circular curvature and of uniform section, under uniformly distributed load.*

Adopting the following notation :

l the span between fixed supports,

w weight intensity per horizontal unit of length l,

A, B, the horizontal and vertical reactions respectively at an abutment,

$\frac{1}{2}Ql$ the moment due to fixture at the abutment $=Q\rho\sin\alpha$,

ρ the mean radius of curvature,

α the angle subtended at the centre of curvature by the half-span,

E the modulus of elasticity of the material of the rib.

Also at any section of the rib taken normal to the curvature,

$x_1\, y_1$ the co-ordinates of its middle point on the neutral axis, from an origin A, before deflexion,

$x_2\, y_2$ corresponding co-ordinates after deflexion,

$\theta_1\, \theta_2$ inclinations to verticality of the normal section, before and after deflexion,

N normal stress on the section,

T and V, thrust and vertical stress on the section,

H couple of lateral stress or bending moment,

M couple of resistance of the section.

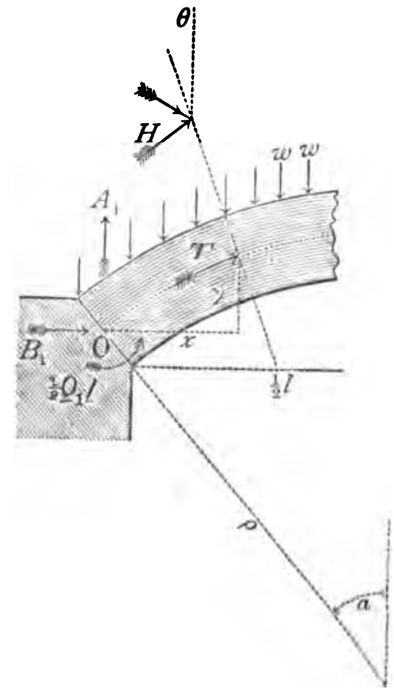


FIGURE 1.

The equation of stress and strain at any section of the rib is $H=M$.

$H=\frac{1}{2}wx^2-Ax+By+\frac{1}{2}Ql$. See Stress, page 38, and $A=\frac{1}{2}wl=w\rho\sin\alpha$,

$H=\rho\{Q\sin\alpha-w\rho\sin\alpha(\sin\alpha-\sin\theta_1)+B(\cos\theta_1-\cos\alpha)+\frac{1}{2}w\rho(\sin\alpha-\sin\theta_1)^2\}$.

Also $M=\frac{EI}{\rho}\cdot\frac{(\delta\theta_2-\delta\theta_1)}{\delta\theta_1}$. See Strains, page 106.

Equating these two values and integrating

$$\theta_2-\theta_1=\frac{\rho^2}{EI}\Big[\{Q\sin\alpha-B\cos\alpha-\tfrac{1}{2}w\rho\sin^2\alpha+\tfrac{1}{4}w\rho\}\theta_1+B\sin\theta_1-\tfrac{1}{4}w\rho\sin\theta_1.\cos\theta_1\Big];$$

any constant would be zero, because when $\theta_1=0$, $\theta_2-\theta_1=0$. Also since when $\theta_1=\alpha$, $\theta_2-\theta_1=0$, we have

$$Q\sin\alpha=B\Big(\cos\alpha-\frac{\sin\alpha}{\alpha}\Big)+\tfrac{1}{4}w\rho\Big(-1+2\sin^2\alpha+\frac{\sin\alpha\cos\alpha}{\alpha}\Big);$$

$$\therefore\ \theta_2-\theta_1=\frac{\rho^2}{EI}\Big[\Big\{\tfrac{1}{4}w\rho.\frac{\sin\alpha\cos\alpha}{\alpha}-B\frac{\sin\alpha}{\alpha}\Big\}\theta_1+B\sin\theta_1-\tfrac{1}{4}w\rho.\sin\theta_1\cos\theta_1\Big].$$

Multiplying this equation by $\rho\delta\theta_1\sin\theta_1$, and by $\rho\delta\theta_1\cos\theta_1$, respectively, and integrating, so as to obtain $x'=x_2-x_1$, and $y'=y_1-y_2$; also changing the notation of θ_1 to θ for convenience, we have

$$x'=\frac{\rho^3}{EI}\Big\{\Big(\tfrac{1}{4}w\rho.\sin\alpha\cos\alpha-B\sin\alpha\Big)\Big(\frac{\sin\theta-\theta\cos\theta}{\alpha}\Big)+\tfrac{1}{2}B(\theta-\sin\theta\cos\theta)-\tfrac{1}{12}w\rho\sin^3\theta\Big\};$$

where any constant would be zero, as $x'=0$, when $\theta=0$.

Also $$y'=\frac{\rho^3}{EI}\cdot\Big\{-\Big(\tfrac{1}{4}w\rho\sin\alpha\cos\alpha-B\sin\alpha\Big)\Big(\frac{\cos\theta+\theta\sin\theta}{\alpha}\Big)+\tfrac{1}{2}B\sin^2\theta-\tfrac{1}{12}w\rho\cos^3\theta+C_1\Big\};$$

for which the value of C_1 will be found subsequently.

As $x'=0$, when $\theta=\alpha$, we will first obtain B.

$$B=\frac{\frac{1}{2}w\rho\sin\alpha\left\{1-\frac{\sin\alpha\cos\alpha}{\alpha}-\frac{2}{3}\sin^2\alpha\right\}}{\alpha+\sin\alpha\cos\alpha-\frac{2\sin^2\alpha}{\alpha}};$$

and this value may be introduced in the expression for $\frac{1}{2}Ql$.

Now to obtain C_1; since $y'=0$, when $\theta=\alpha$,

$$C_1=\left(\frac{1}{4}w\rho.\frac{\sin\alpha\cos\alpha}{\alpha}-B\frac{\sin\alpha}{\alpha}\right)\left(\cos\alpha+\alpha\sin\alpha\right)+\frac{1}{2}B\sin^2\alpha+\frac{1}{12}w\rho\cos^3\alpha;$$

in which the value of B may be introduced.

The flexure, or greatest value of y', will be obtained from the general expression for y', when $\theta=0$.

Thus we now have in known terms the values of A, B, $\frac{1}{2}Ql$, x', y', H, and M.

Also as $V=A-wx$; $N=B\cos\theta+V\sin\theta$; $T=B\sin\theta-V\cos\theta$; the values of N and T are known.

The strain on either flange member separately, as M_1 and M_2, may be obtained by putting d=effective depth of the rib, or distance between the centres of strain of the two flange members (which may be calculated in some cases and merely approximately determined in others). Then $M_1=-\frac{M}{d}$ for the upper; $M_2=+\frac{M}{d}$ for the lower; the positive values in either M_1 or M_2 denoting compressive, and negative values tensile strains.

Solution Number 2.—Fixed rib of circular curvature under uniformly distributed load, being of uniform section throughout the greater part of its length, but having besides a greater section also uniform for a short distance at each end.

Adopting the notation of the last solution with the following additions,

I_1 the moment of inertia of the end section,

I_2 „ „ „ middle section,

α, angle subtended at the centre of curvature by the half arc of the complete rib,

β, angle similarly subtended by the half arc of the portion of rib having the small section.

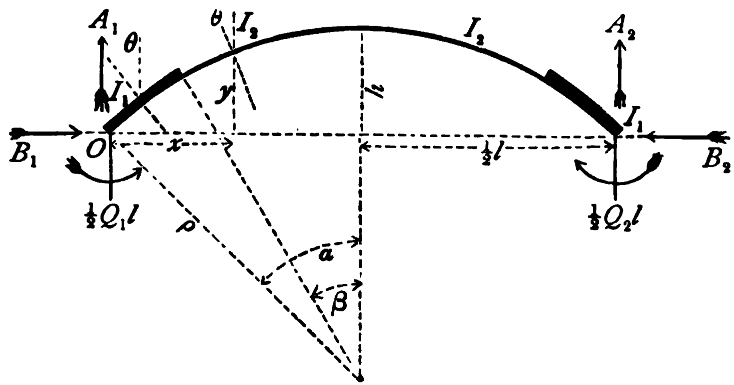


FIGURE 2.

The general equation of stress and strain at any section of the rib is $H=M$, as in the last solution; but two distinct cases will occur in each symmetrical half rib, as M will differ with each section. The two results correspond generally to the single case of the last solution, being (1) for the end piece of increased section,

$$\theta_2-\theta_1=\frac{\rho^2}{EI_1}\left\{(Q\sin\alpha-B\cos\alpha-\tfrac{1}{2}w\rho^2\sin\alpha+\tfrac{1}{4}w\rho)\theta_1+B\sin\theta_1-\tfrac{1}{4}w\rho\sin\theta_1\cos\theta_1+C_1\right\} \quad . \quad . \quad . \quad . \quad (1)$$

and (2) for the middle piece of ordinary section,

$$\theta_2-\theta_1=\frac{\rho^2}{EI_2}\left\{(Q\sin\alpha-B\cos\alpha-\tfrac{1}{2}w\rho\sin^2\alpha+\tfrac{1}{4}w\rho)\theta_1+B\sin\theta_1-\tfrac{1}{4}w\rho\sin\theta_1\cos\theta_1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In equation (1), $\theta_1=\alpha$, $\theta_2-\theta_1=0$; whence the value of C_1 may be obtained. In equation (2) any constant would be zero, as when $\theta_1=0$, $\theta_2-\theta_1=0$. Also when $\theta=\beta$, the two equations (1) and (2) become identical. Hence

$$C_1=-(Q\sin\alpha-B\cos\alpha-\tfrac{1}{2}w\rho\sin^2\alpha+\tfrac{1}{4}w\rho)\alpha-B\sin\alpha+\tfrac{1}{4}w\rho\sin\alpha\cos\alpha.$$

Also

$$[I_2\alpha+(I_1-I_2)\beta].\{Q\sin\alpha-B\cos\alpha-\tfrac{1}{2}w\rho\sin^2\alpha+\tfrac{1}{4}w\rho\}+B[I_2\sin\alpha+(I_1-I_2)\sin\beta]-\tfrac{1}{4}w\rho[I_2\sin\alpha\cos\alpha+(I_1-I_2)\sin\beta\cos\beta]=0.$$

Adopting now some additional symbols for convenience,

$$\sigma_1=\frac{I_2}{I_1}.\alpha+\frac{I_1-I_2}{I_1}.\beta; \qquad \sigma_4=\frac{I_2}{I_1}.\tfrac{1}{6}\sin^3\alpha+\frac{I_1-I_2}{I_1};$$

$$\sigma_2=\frac{I_2}{I_1}.\sin\alpha\cos\alpha+\frac{I_1-I_2}{I_1}.\sin\beta\cos\beta; \quad \sigma_5=\sigma_1+\sigma_2-\frac{2\sigma_3^{\,2}}{\sigma_1};$$

$$\sigma_3=\frac{I_2}{I_1}.\sin\alpha+\frac{I_1-I_2}{I_1}.\sin\beta; \qquad \sigma_6=\tfrac{1}{4}\frac{\sigma_3}{\sigma_1}\Big(\sigma_1-\sigma_2\Big)-\sigma_4;$$

then $Q\sin\alpha=B\Big(\cos\alpha-\dfrac{\sigma_3}{\sigma_1}\Big)+\tfrac{1}{4}w\rho\Big(-1+2\sin^2\alpha+\dfrac{\sigma_2}{\sigma_1}\Big)$;

and introducing this in equations (1) and (2) they become

$$\theta_2-\theta_1=\frac{\rho^2}{EI_1}\Big\{\Big(\tfrac{1}{4}w\rho.\frac{\sigma_2}{\sigma_1}-B.\frac{\sigma_3}{\sigma_1}\Big)\theta_1+B\sin\theta_1-\tfrac{1}{4}w\rho\sin\theta_1\cos\theta_1+C_1\Big\} \quad . \quad . \quad . \quad (3)$$

$$\theta_2-\theta_1=\frac{\rho^2}{EI_2}\Big\{\Big(\tfrac{1}{4}w\rho.\frac{\sigma_2}{\sigma_1}-B.\frac{\sigma_3}{\sigma_1}\Big)\theta_1+B\sin\theta_1-\tfrac{1}{4}w\rho\sin\theta_1\cos\theta_1\Big\} \quad . \quad . \quad . \quad (4)$$

and $\quad C_1=-\Big(\tfrac{1}{4}w\rho.\dfrac{\sigma_2}{\sigma_1}-B.\dfrac{\sigma_3}{\sigma_1}\Big)\alpha-B\sin\alpha+\tfrac{1}{4}w\rho\sin\alpha\cos\alpha.$

Multiplying equations (3) and (4) by $\rho\delta\theta_1\sin\theta_1$, and integrating to obtain $x'=x_2-x_1$; and now denoting θ_1 as θ simply, we obtain,

$$x'=\frac{\rho^3}{EI_1}\left\{\left(\tfrac{1}{4}w\rho\frac{\sigma_2}{\sigma_1}-B\frac{\sigma_3}{\sigma_2}\right)\left(\sin\theta-\theta\cos\theta\right)+\tfrac{1}{2}B(\theta-\sin\theta\cos\theta)-\tfrac{1}{12}w\rho\sin^3\theta+C_1\cos\theta+C_2\right\}; \quad . \quad . \quad . \quad (5)$$

$$x'=\frac{\rho^3}{EI_2}\left\{\left(\tfrac{1}{4}w\rho\frac{\sigma_2}{\sigma_1}-B.\frac{\sigma_3}{\sigma_2}\right)\left(\sin\theta-\theta\cos\theta\right)+\tfrac{1}{2}B(\theta-\sin\theta\cos\theta)-\tfrac{1}{12}w\rho\sin^3\theta\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In equation (6) the constant would be zero, as $x'=0$ when $\theta=0$. In equation (5) C_2 must be found from $x'=0$, when $\theta=\alpha$. Also equations (5) and (6) are identical when $\theta=\beta$. Hence

$$C_2=-\left(\tfrac{1}{4}w\rho\frac{\sigma_2}{\sigma_1}-B\frac{\sigma_3}{\sigma_1}\right)\sin\alpha-\tfrac{1}{2}B\left(\alpha+\sin\alpha\cos\alpha\right)+\tfrac{1}{12}w\rho(3\sin\alpha-2\sin^3\alpha);$$

and $\left(I_1-I_2\right)\left\{\left(\tfrac{1}{4}w\rho\frac{\sigma_2}{\sigma_1}-B\frac{\sigma_3}{\sigma_1}\right)\left(\sin\beta-\beta\cos\beta\right)+\tfrac{1}{2}\beta\left(\beta-\sin\beta\cos\beta\right)-\tfrac{1}{12}w\rho\sin^3\beta\right\}+I_2\left\{\left(\tfrac{1}{4}w\rho\frac{\sigma_2}{\sigma_1}-B\frac{\sigma_3}{\sigma_1}\right)\right.$
$\left.\left(\sin\alpha-\alpha\cos\alpha\right)+\tfrac{1}{2}\beta(\alpha+\sin\alpha\cos\alpha-2\sin\alpha\cos\beta)\right\}+I_2w\rho.\tfrac{1}{12}(3\sin\alpha\cos\alpha\cos\beta-3\sin\alpha+2\sin^3\alpha)=0;$

whence $B=2w\rho.\dfrac{\sigma_6}{\sigma_5}$; according to notation.

Introducing this in the former value of $\tfrac{1}{2}Ql$;

$$Q\sin\alpha=w\rho^2\left\{-\frac{2\sigma_6}{\sigma_5}.\left(\frac{\sigma_3}{\sigma_1}-\cos\alpha\right)+\tfrac{1}{4}\left(-1+2\sin^2\alpha+\frac{\sigma_2}{\sigma_1}\right)\right\}.$$

Now, multiplying equations (3) and (4) by $\rho.\delta\theta_1 \cos\theta_1$, and integrating to obtain $y'=y_1-y_2$, and using θ for θ_1 in the notation,

$$y'=\frac{\rho^3}{EI_1}\left\{-\left(\tfrac{1}{4}w\rho.\frac{\sigma_2}{\sigma_1}-B.\frac{\sigma_3}{\sigma_1}\right)\left(\cos\theta+\theta\sin\theta\right)-\tfrac{1}{2}B\sin^2\theta-\tfrac{1}{12}w\rho\cos^3\theta-C_1\sin\theta+C_3\right\}; \quad . \quad . \quad . \quad . \quad (7)$$

$$y'=\frac{\rho^3}{EI_2}\left\{-\left(\tfrac{1}{4}w\rho.\frac{\sigma_2}{\sigma_1}-B.\frac{\sigma_3}{\sigma_1}\right)\left(\cos\theta+\theta\sin\theta\right)-\tfrac{1}{2}B\sin^2\theta-\tfrac{1}{12}w\rho\cos^3\theta+C_4\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In equation (7), C_3 may be found from $y'=0$, when $\theta=\alpha$. Also equations (7) and (8) are identical when $\theta=\beta$. Hence

$$C_3=\left(\tfrac{1}{4}w\rho.\frac{\sigma_2}{\sigma_1}-B\frac{\sigma_3}{\sigma_1}\right)\cos\alpha-\tfrac{1}{2}B\sin^2\alpha+\tfrac{1}{12}w\rho(3\cos\alpha-2\cos^3\alpha);$$

$$C_4=\frac{I_2}{I_1}.\left(C_3-C_1\sin\beta\right)+\frac{I_1-I_2}{I_1}.\left\{\left(\tfrac{1}{4}w\rho.\frac{\sigma_2}{\sigma_1}-B.\frac{\sigma_3}{\sigma_1}\right)\left(\cos\beta+\beta\sin\beta\right)+\tfrac{1}{2}B\sin^2\beta+\tfrac{1}{12}w\rho\cos^3\beta\right\};$$

in which the values of C_1, C_3, and B may be introduced.

Hence the values of A, B, $\tfrac{1}{2}Ql$, H, and M are now expressed in known terms.

Also as $V=A-wx$; $N=B\cos\theta+V\sin\theta$; $T=B\sin\theta-V\cos\theta$; the values of N and T are known; also the flexure which is the value of y' when $\theta=0$.

The strains on the upper and the lower members, M_1 and M_2, are obtained from M.

$M_1=-\dfrac{M}{d}$; and $M_2=+\dfrac{M}{d}$; where d is the effective depth of the curved rib; positive signs denoting compression, and negative signs tension.

Solution Number 3.—Fixed rib of circular curvature and uniform section under partial load.

As this rib has been treated under a uniform load, in Solution No. 1, the notation there adopted will apply here, with some additions.

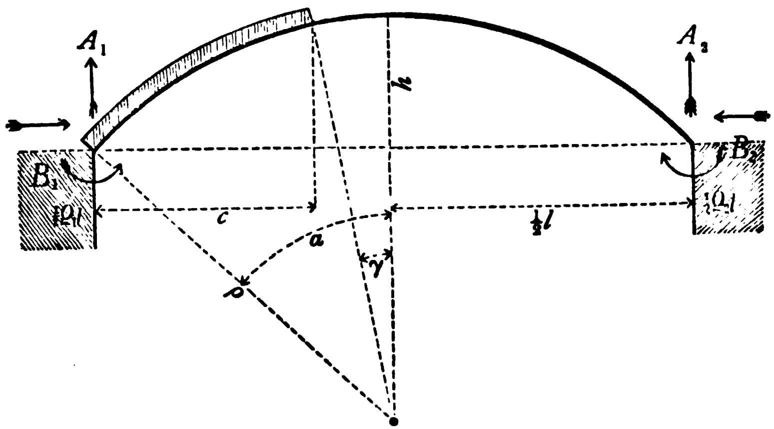


FIGURE 3.

Let the partial load extend over less than half the rib, as in the figure, so that c is less than $\frac{1}{2}l$, and that both γ and α may be estimated on the same side of the vertical bisecting the rib: then $c=\rho(\sin\alpha-\sin\gamma)$; and the partial load $=wc=w\rho(\sin\alpha-\sin\gamma)$.

The equation of stress and strain at any section of the rib is generally $H=M$; but in the left half-rib to be first treated there are two distinct cases, one in the loaded arc, the other in the unloaded arc.

In (1), $x=0$ to c; $H=\frac{1}{2}Ql+\frac{1}{2}wx^2-Ax+By$;

In (2), $x=c$ to l; $H=\frac{1}{2}Ql-Ax+By+wc(x-\frac{1}{2}c)$;

and everywhere $M=\dfrac{EI}{\rho}\cdot\dfrac{\delta\theta_2-\delta\theta_1}{\delta\theta_1}$, as in Strains, p. 106; and as in Solution No. 1, p. 217.

Equating these values and integrating, we have (1) in the loaded arc and (2) in the unloaded arc, as follow:

$$\theta_2-\theta_1=\frac{\rho^2}{EI}\Big\{(Q\sin\alpha-A\sin\alpha-B\cos\alpha+\tfrac{1}{2}w\rho\sin^2\alpha+\tfrac{1}{4}w\rho)\theta_1-(A-\tfrac{1}{2}wl)\cos\theta_1+B\sin\theta_1+\tfrac{1}{4}w\rho\sin\theta_1.\cos\theta_1+C_1\Big\}; \quad (1)$$

$$\theta_2-\theta_1=\frac{\rho^2}{EI}\Big\{(Q\sin\alpha-A\sin\alpha-B\cos\alpha+\tfrac{1}{2}w\rho\sin^2\alpha-w\rho.\sin^2\gamma)\theta_1-(A-\tfrac{1}{2}wl+w\rho\sin\gamma)\cos\theta_1+B\sin\theta_1+C_2\Big\} \quad (2)$$

The rib being fixed at both ends, in equation (1) $\theta_2-\theta_1=0$, when $\theta=+\alpha$; and in equation (2) $\theta_2-\theta_1=0$, when $\theta=-\alpha$; hence, half their difference will then be zero. Also, equations (1) and (2) are identical when $\theta=\gamma$; hence, half their difference will then yield the value of $\frac{1}{2}(C_1-C_2)$. Hence

$$\{Q\sin\alpha-A\sin\alpha=B\cos\alpha+\tfrac{1}{2}w\rho\sin^2\alpha+\tfrac{1}{8}w\rho-\tfrac{1}{4}w\rho.\sin^2\gamma\}.\alpha+\tfrac{1}{2}w\rho\sin\gamma.\cos\alpha-\tfrac{1}{8}w\rho\sin\alpha\cos\alpha+B\sin\alpha+\tfrac{1}{2}(C_1-C_2)=0;$$

and $\frac{1}{2}(C_1-C_2)=\frac{1}{8}w\rho(\gamma+2\gamma\sin^2\gamma+3\sin\gamma\cos\gamma)$;

$$=\tfrac{1}{8}w\rho.\xi_1\text{; for brevity} \quad (3)$$

The value of $\frac{1}{2}(C_1+C_2)$ will be afterwards obtained from the same condition, when there is opportunity for further reduction.

Introducing the value of $\frac{1}{2}(C_1-C_2)$ in the former long equation, we obtain

$$Q\sin\alpha=A\sin\alpha+B\cos\alpha+w\rho\Big(-\tfrac{1}{8}-\tfrac{1}{2}\sin^2\alpha+\frac{\sin\alpha\cos\alpha}{8\alpha}\Big)-B.\frac{\sin\alpha}{\alpha}+w\rho\Big(\tfrac{1}{4}\sin^2\gamma-\frac{\cos\alpha\sin\gamma}{2\alpha}+\frac{\xi_1}{8\alpha}\Big) \quad (4)$$

Using for brevity the following symbols in the value of $Q \sin \alpha$, before introduction in equations (1) and (2),

$$\Gamma = -B . \frac{\sin \alpha}{\alpha} + \frac{w\rho}{8\alpha}\left(-4 \cos \alpha \sin \gamma + \xi_1\right);$$

$\varDelta = \frac{1}{8} w\rho (1 - 2 \sin^2 \gamma)$; then we have

$$\theta_2 - \theta_1 = \frac{\rho^2}{EI}\left\{(\Gamma + \varDelta)\theta_1 - (A - \tfrac{1}{2} wl) . \cos \theta_1 + B \sin \theta_1 - \tfrac{1}{4} w\rho \sin \theta_1 \cos \theta_1 + C_1\right\} \quad . \quad . \quad . \quad . \quad (5)$$

$$\theta_2 - \theta_1 = \frac{\rho^2}{EI}\left\{(\Gamma - \varDelta)\theta_1 - (A - \tfrac{1}{2} wl + w\rho \sin \gamma) . \cos \theta_1 + B \sin \theta_1 + C_2\right\} \quad . \quad . \quad . \quad . \quad (6)$$

Multiplying these equations by $\rho \delta \theta_1 \sin \theta_1$ and integrating, we obtain the two corresponding equations for $x' = x_2 - x_1$, the horizontal displacements in the loaded and unloaded arcs.

$$x' = \frac{\rho^3}{EI}\left\{(\Gamma + \varDelta)(\sin \theta - \theta \cos \theta) - (A - \tfrac{1}{2} wl)\tfrac{1}{2} \sin^2 \theta + \tfrac{1}{2} B(\theta - \sin \theta \cos \theta) - \tfrac{1}{12} w\rho \sin^3 \theta - C_1 \cos \theta + C_3\right\} \quad (7)$$

$$x' = \frac{\rho^3}{EI}\left\{(\Gamma - \varDelta)(\sin \theta - \theta \cos \theta) - (A - \tfrac{1}{2} wl + w\rho \sin \gamma) . \tfrac{1}{2} \sin^2 \theta + \tfrac{1}{2} B(\theta - \sin \theta \cos \theta) - C_2 \cos \theta + C_4\right\} \quad . \quad (8)$$

As the rib is fixed at both ends, we have in equation (7), $x' = 0$, when $\theta = +\alpha$; and in equation (8) we have $x' = 0$, when $\theta = -\alpha$; hence half their difference will then be zero. Also equations (7) and (8) will be identical when $\theta = \gamma$; hence half their difference will then yield the value of $\frac{1}{2}(C_3 - C_4)$.

Q

$\therefore\ \Gamma(\sin\alpha-\alpha\cos\alpha)+\frac{1}{4}w\rho.\sin\gamma.\sin^2\alpha+\frac{1}{2}B(\alpha-\sin\alpha.\cos\alpha)-\frac{1}{12}w\rho\sin^3\alpha-\frac{1}{2}(C_1-C_2)+\frac{1}{2}(C_3-C_4)=0\,;$

also

$$\tfrac{1}{2}(C_3-C_4)=-\Gamma(\sin\gamma-\cos\gamma)-\tfrac{1}{4}w\rho\sin^3\gamma+\tfrac{1}{24}w\rho\sin^3\gamma+\tfrac{1}{2}(C_1-C_2).\cos\gamma\,;\ =-\tfrac{1}{2}w\rho(\sin\gamma+\tfrac{1}{6}\sin^3\gamma)$$
$$=-\tfrac{1}{2}w\rho.\xi_2 \text{ for brevity} \quad . \quad . \quad . \quad (9)$$

The value of $\frac{1}{2}(C_3+C_4)$ will be afterwards similarly obtained. Introducing (9) and (3) in the former long equation, supplying the value of T, and reducing we obtain

$$B=\frac{w\rho}{4(\alpha^2+\alpha\sin\alpha\cos\alpha-2\sin^2\alpha}\left\{4\sin\alpha\sin\beta(\cos\alpha+\tfrac{1}{2}\alpha\sin\alpha)+\sin\alpha.\xi_1+\sin\alpha(\alpha-\sin\alpha\cos\alpha)-\tfrac{2}{3}(\sin^3\alpha-\sin^3\beta)\right\} \quad . \quad . \quad (10)$$

Proceeding now to the vertical deflexions; multiplying equations (5) and (6) by $\rho\delta\theta_1\cos\theta_1$ and integrating, we obtain the corresponding two equations $y'=y_1-y_2$ in the loaded and in the unloaded arcs; putting also θ for θ_1, we have

$$y'=\frac{\rho^3}{EI}.\left\{-(\Gamma+\Delta)(\cos\theta+\theta\sin\theta)+\tfrac{1}{2}(A-\tfrac{1}{2}wl)(\theta+\sin\theta.\cos\theta)-\tfrac{1}{2}B\sin^2\theta-\tfrac{1}{12}w\rho\cos\theta-C_1\sin\theta+C_5\right\}.\ (11)$$

$$y'=\frac{\rho^3}{EI}\left\{-(\Gamma-\Delta)(\cos\theta+\theta\sin\theta)+\tfrac{1}{2}(A-\tfrac{1}{2}wl+w\rho\sin\gamma).(\theta+\sin\theta\cos\theta)-\tfrac{1}{2}B\sin^2\theta-C_2\sin\theta+C_6\right\}\ (12)$$

These two equations being identical, when $\theta=\gamma$, we then obtain $\frac{1}{2}(C_5-C_6)$ by halving their difference,

$$\tfrac{1}{2}(C_5-C_6)=\Delta(\cos\gamma+\gamma\sin\gamma)+\tfrac{1}{4}w\rho\sin\gamma(\gamma+\sin\gamma\cos\gamma)+\tfrac{1}{24}w\rho\cos^3\gamma+\tfrac{1}{2}(C_1-C_2).\sin\gamma;$$
$$=\tfrac{1}{4}w\rho(\cos\gamma+\gamma\sin\gamma-\tfrac{1}{3}\cos^3\gamma);\ =\tfrac{1}{2}w\rho\,\xi_3 \text{ for brevity} \quad . \quad . \quad . \quad (13)$$

Also from the fixity of the rib at both ends, in equation (11), $y'=0$ when $\theta=a$; and in equation (12), $y'=0$ when $\theta=a$; hence half their difference will be zero under those conditions.

Also we can obtain A in known terms, after introducing the values of Δ and of the differences of the constants. This when reduced is

$$A=\frac{w\rho}{a-\sin a\cos a}\cdot\left\{\tfrac{1}{2}\sin^2\gamma(\cos a-a+\sin a\cos a)-\xi_3+\tfrac{1}{2}\cos a(1-\tfrac{1}{3}\cos^2 a)\right\}+w\rho\sin a \quad . \quad . \quad . \quad . \quad (14)$$

Having now A and B in known terms, the value of $Q\sin a$ in equation (4) may be similarly reduced. Whence also the values of H and of M can be explicitly expressed.

But to evaluate x' and y', the whole of the constants must be found. Referring to equations (3), (9), and (13), we may evidently derive the half sums of the pairs of constants from the same conditions that afforded their half differences. Having obtained them, they may thus be expressed in known terms,

$$\tfrac{1}{2}(C_1+C_2)=-\Delta a+(A-\tfrac{1}{2}wl+\tfrac{1}{2}w\rho\sin\gamma)\cos a+\tfrac{1}{16}w\rho\sin 2a;$$

$$\tfrac{1}{2}(C_3+C_4)=-\Delta_2\sin a(A-\tfrac{1}{2}wl+\tfrac{1}{2}w\rho\sin\gamma)(1-\tfrac{1}{2}\sin^2 a)+\tfrac{1}{8}w\rho(\sin a-\tfrac{2}{3}\sin^3 a);$$

$$\tfrac{1}{2}(C_5+C_6=-\frac{B}{2a}\left(\sin 2a+a\sin^2 a\right)+\frac{w\rho.\cos a}{8a}.\xi_1+\tfrac{1}{4}w\rho\sin\gamma\left(a-\sin a\cos a-\frac{2\cos^3 a}{a}\right)+\tfrac{1}{8}w\rho\left\{\frac{\sin a\cos^2 a}{a}+\cos a-\tfrac{2}{3}\cos^3 a\right\};$$

whence the values of the separate constants can be obtained.

If the reactions at the right abutment are required as well as those on the left abutment, they may be reduced from the following:

$$A_2 = wc - A_1;\ B_2 = B_1;\ \tfrac{1}{2}Q_2 l = -\tfrac{1}{2}Q_1 l + Al - wc.(l - \tfrac{1}{2}c).$$

The method adopted in this solution is that of Chauvenet, applied by him to a more complicated case, combining both partial load and varied section.

This and the last solution treat these conditions separately. But if it is desired to combine them there appears no need for an express solution from the commencement; as the last Solution No. 2 may be adopted, and to the results a modification due to partial loading combined with equally distributed uniform load may be applied.

See remarks in Fifth Case of Solution Number 6 of Curved Ribs, page 238.

Solution Number 4.—Temperature strains due to the expansion of a fixed curved rib of circular curvature and uniform section.

The effect of increase of temperature on a fixed metallic curved rib is to add a fresh set of stresses to those due to equally distributed permanent load and moving load. The sums and differences of all these stresses may be required in any complete investigation of a rib, as well as of the piers and abutments supporting it.

Adopting the symbols and general method of Solution Number 1, for a rib of uniform section under equally distributed load, also let l_1 be the linear elongation or stretch of material due to the extreme possible change of temperature.

If the rib were straight and freely supported, the ex-

panded length of span would be $l(1+l_1)$, and the horizontal displacement at each end $\frac{1}{2}ll_1$; also as precision in expansions of this sort and on this scale is unattainable, the approximative result of treating a curved rib as straight will be sufficiently correct in most cases even with large spans. Also, though the condition of fixity prevents the span from enlarging, the enlargement of the corresponding free span is the condition affecting the stresses resulting from expansion in a fixed span.

Since the load and vertical stresses are unaffected by change of temperature, the moments at any section are

$$H=By+\tfrac{1}{2}Ql\text{; and } M=\frac{EI}{\rho}\cdot\frac{\partial\theta_2-\partial\theta_1}{\partial\theta_1};$$

also $H=M$ at every section.

$$\therefore\ \partial\theta_2-\partial\theta_1=\frac{\rho^2}{EI}\Big\{Q\sin\alpha+B(\cos\theta_1-\cos\alpha)\Big\};$$

$$\therefore\ \theta_2-\theta_1=\frac{\rho^2}{EI}\Big\{(Q\sin\alpha-B\cos\alpha)\theta_1+B\sin\theta_1\Big\};$$

where any constant is zero, since when $\theta_1=0$, $\theta_2-\theta_1=0$; and we may obtain $Q\sin\alpha$ from the condition that $\theta_2-\theta_1=0$, when $\theta_1=\alpha$.

$$\text{Hence } Q\sin\alpha=B\Big(\cos\alpha-\frac{\sin\alpha}{\alpha}\Big);$$

$$\text{and } \theta_2-\theta_1=\frac{\rho^2}{EI}\Big\{B\sin\theta_1-B\theta_1.\frac{\sin\alpha}{\alpha}\Big\};$$

obtaining from this the displacements and deflexions in the same way as in the former solutions, we have $x'=x_2-x_1$;

$$x'=\frac{\rho^3}{EI}.B.\Big\{\tfrac{1}{2}(\theta_1-\sin\theta\cos\theta)-\frac{\sin\alpha}{\alpha}\Big(\sin\theta-\theta\cos\theta\Big)\Big\};$$

where any constant is zero, as when $\theta_1=0$, $x'=0$.

$$\text{Also } y'=\frac{\rho^3}{EI}.B\Big\{-\tfrac{1}{2}\sin^2\theta+\frac{\sin\alpha}{\alpha}\Big(\cos\theta+\theta\sin\theta\Big)+C_1\Big\};$$

where C_1 can be found, as when $\theta=\alpha$, $y=0$. But as $x'=\frac{1}{2}l.l_1$, when $\theta=\alpha$, we obtain through this condition

$$B=\frac{l.l_1.EI}{\rho^3}.\left\{\frac{\alpha}{\alpha^2+\alpha\sin\alpha.\cos\alpha-2\sin^2\alpha}\right\}.$$

Introducing this value of B into the expression for $\frac{1}{2}Ql$, the latter becomes known; and by further substitution the value of H or M is also known; the deflexion at any section can therefore be calculated.

The thrust and the shearing stress at any section will be respectively $B\cos\theta$ and $B\sin\theta$.

General Solution.

Solution Number 5.—*Theory of Curved Ribs of any curvature and of unknown pressure-curve.*

Let the curvature of the rib be of any sort determined by known relations of x and y, the rectangular co-ordinates of the curve from an origin O at one support, so that the conditions of curvature may be applied afterwards in any special case. Let it be assumed that the position of the

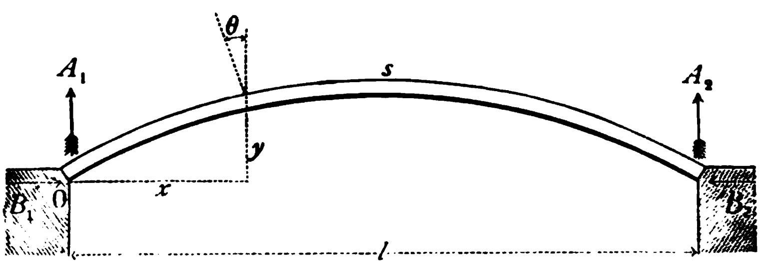


FIGURE 4.

pressure-curve in the rib-section at the crown is entirely unknown, and that it may not be arrived at by repeated approximation, or be located within any known limits, such as that of the effective depth of the rib-section.

Treating now the curve of the neutral lamina of the rib; its rise h at midspan is unknown, its span l is given, and its length of arc s is dependent on the yet undeclared relations of x and y; the relations of the differentials ∂x, ∂y, and ∂s as well as those of small finite differences Δx, Δy, and Δs are known, the former with exactitude.

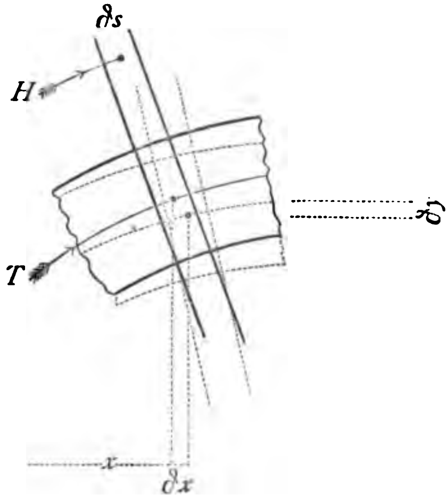


FIGURE 5.

Taking an elementary portion of arc ∂s, acted on by a single force applied anywhere on the axis normal to ∂s, the horizontal displacement of the centre of gravity of the section there may be represented by $\partial x'$ or by $\Delta x'$. The single force may be resolved into a thrust T and a couple, or bending moment Hy, acting at any distance y from the centre of gravity of the section (see also page 37), while the shearing stress parallel to the section may be neglected as comparatively ineffective.

Then if

E be the elasticity of the material,

S the sectional area at this section,

I its moment of inertia about a horizontal cross axis passing through its centre of gravity,

$$\partial x' \text{ or } \Delta x' = \frac{H}{EI} \cdot y \partial s + \frac{T}{ES} \cdot \partial x.$$

The total horizontal displacement over the whole of the curved rib, due to a series of such single forces similarly

resolved and acting at sections whose typical areas and moments of inertia are S and I, will hence be

$$x' = \int \frac{H}{EI} y \cdot \frac{ds}{dx} . dx - \int \frac{T}{ES} . dx.$$

Also the total horizontal displacement over the whole rib, due to stress from change of temperature (and correspondingly also from wedging) can be expressed in the form $x''=l.l'$; where l' is the elongation. So that under all causes combined, $x'+x''$ is the whole horizontal displacement.

But as the span is of invariable length, we have $x'+x''=0$; therefore

$$E.l.l' = \int \left\{ \frac{Hy}{I} . \frac{ds}{dx} + \frac{T}{S} \right\} . dx = 0 \quad . \quad . \quad . \quad \text{(I.)}$$

the fundamental equation, through which an unknown horizontal stress at one abutment, such as either B_1 or B_2 in the figure, may be arrived at in known terms.

The single forces, typified by W, may evidently be direct weight applied vertically or radially, or in any direction, and their disposition may vary at different parts of the arc, or be continuous according to any special mode; either per unit of length of arc or per unit of length of span. Also the section, and the sectional values S and I, may vary at different parts of the rib. All such parts may be similarly treated, although the values will require separate determination. The summation expressed in the equation (1) may be either effected in continuous parts through the integration of differentials, or in discontinuous parts through simple summation by quadrature of finite differences.

The general equations of static resolution of stress give with vertical loading $A_1+A_2=\Sigma W$; $A_1 l=c\Sigma W$; and the moment of one abutment reaction B_2 from an equation of

moments. But the other reaction B_1 requires the aid of the fundamental equation (1).

Putting $H_1 = H_2 - B_1 y$; and $T_1 = T_2 - B_1 . \frac{\delta x}{\delta s}$; H_2 and T_2 being free from B_1 are known; and we obtain

$$B_1 = \frac{EI . l' + \int_0^l \left\{ \frac{H_2 y}{I} . \frac{\delta s}{\delta x} + \frac{T_2}{S} \right\} . \delta x}{\int_0^l \left\{ \frac{y^2}{I} . \frac{\delta s}{\delta x} + \frac{1}{S} . \frac{\delta x}{\delta s} \right\} . \delta x .} ; \quad . \quad . \quad (\text{II.})$$

where the term $\int_0^l \frac{1}{S} . \frac{\delta x}{\delta s} . \delta x$ is of comparatively small value.

In evaluating this expression, with a series of divisions of arc or of length of span, it may be noticed that the values of H_2, partial bending moments independent of y, are those of the bending moments of a straight horizontal girder at corresponding abscissæ x. These abscissæ may or may not require calculation according as the series of divisions occur along S or along l. The values of H_2 are, however, merely used as intermediate quantities or auxiliaries for arriving at the values of H_1, the complete bending moments; so that there is not any assumption of uniform stiffness of rib.

The thrusts T_2 are compressive and of negative sign; and when $x < \frac{1}{2} l$, the values of θ to the other side of a vertical axis drawn through the crown are considered negative.

If the equal divisions be taken along the span, as δx or Δx; and θ be the inclination of any section to verticality, so that $\frac{\delta x}{\delta s} = \cos \theta$; equation (II.) becomes

$$B_1 = \frac{EIl' + \int_0^l \left\{ \frac{H_2 y}{I \cos \theta} + \frac{T_2}{S} \right\} . \delta x}{\int_0^l \frac{y^2}{I \cos \theta} . \delta x} \quad . \quad . \quad . \quad (\text{III.})$$

If the equal divisions be taken along the arc as δs or Δs, we then have

$$B_1 = \frac{EIl' + \int_0^s \left\{ \frac{H_2 y}{I} + \frac{T_2 \cos\theta}{S} \right\} . \delta s}{\int_0^s \frac{y^2}{I} . \delta s} \quad . \quad . \quad . \quad \text{(IV.)}$$

Having B_1, H_2, and T_2, the partial bending moments free of expansion and the partial thrust, due only to A and ΣW, the vertical forces; now the value of the complete bending moment and of the complete thrust may be obtained through

$$H_1 = H_2 - B_1 y\,; \text{ and } T_1 = T_2 - B_1 \cos\theta\,; \quad . \quad . \quad . \quad . \quad . \quad \text{(V.)}$$

whence also we get the resistance at the corresponding section, when a is the axial distance of the furthest fibre,

$$R = \frac{H_1 a}{I} - \frac{T_1}{S}\,; \quad . \quad . \quad . \quad . \quad . \quad \text{(VI.)}$$

R being the intensity of strain per superficial unit of section, so that a series of values of R may be obtained as strains induced in the rib at any number of points of division.

The pressure-curve may be traced at any time after H and T are found, as $\frac{H}{T}$ expresses the distance of the centre of pressure from the neutral lamina at any section, this distance being measured above it when H is positive, and below it when H is negative.

This general method of treating the theory of curved ribs is that of Bresse; its application in special solutions might be facilitated by using tabulated values of some of the terms, some of which are very complicated, when partial loading is taken into consideration.

Solution Number 6.—*The abutment reaction in special cases of curved ribs.*

The following values of B_1, the required horizontal stress at the abutment, which is expressed in general terms in the last solution, are here given for special cases; they are all either due to Bresse or derived from his values.

First Case.—Any curved rib, symmetrical both in form and in loading.

(1.) If resting on free supports; taking the axis of y through the crown, and x along the span,

$$B_1=\frac{El.l'+\int_0^{\frac{1}{2}l}\left\{\frac{H_2 y}{I}.\frac{\partial s}{\partial x}+\frac{T_2}{S}\right\}.\partial x}{\int_0^{\frac{1}{2}l}\frac{y^2}{I}.\frac{\partial s}{\partial x}.\partial x.};$$

(2.) If the extremities are built in or fixed, taking the origin at midspan.

Let Q be the moment of fixture retaining a constant inclination at the support, and putting $H=H_8-B_1 y+Q$; the value of Q is detached, and the values of B_1 and of Q are obtained through the two following equations.

$$Q.\int_0^{\frac{1}{2}l}\frac{1}{EI}.\frac{\partial s}{\partial x}.\partial x-B_1\int_0^{\frac{1}{2}l}\frac{y}{EI}.\frac{\partial s}{\partial x}.\partial x+\int_0^{\frac{1}{2}l}\frac{H_8}{EI}.\frac{\partial s}{\partial x}=0;$$

$$B_1\left\{\int_0^{\frac{1}{2}l}\frac{y^2}{EI}.\frac{\partial s}{\partial x}.\partial x+\int_0^{\frac{1}{2}l}\frac{1}{ES}\frac{\partial x}{\partial s}.\partial x\right\}-Q\int_0^{\frac{1}{2}l}\frac{y}{EI}.\frac{\partial s}{\partial x}.\partial x$$

$$=\tfrac{1}{2}l.l'.\int_0^{\frac{1}{2}l}\frac{H_8 y}{EI}.\frac{\partial s}{\partial x}.\partial x+\int_0^{\frac{1}{2}l}\frac{T_2}{ES}.\partial x.$$

Second Case.—A rib of circular curvature and of uniform section resting on free supports, strained by a single force.

(1.) When the force is a single representative weight W acting vertically, and applied at a point on the curve

whose angular position is denoted by θ, the inclination of the section to verticality.

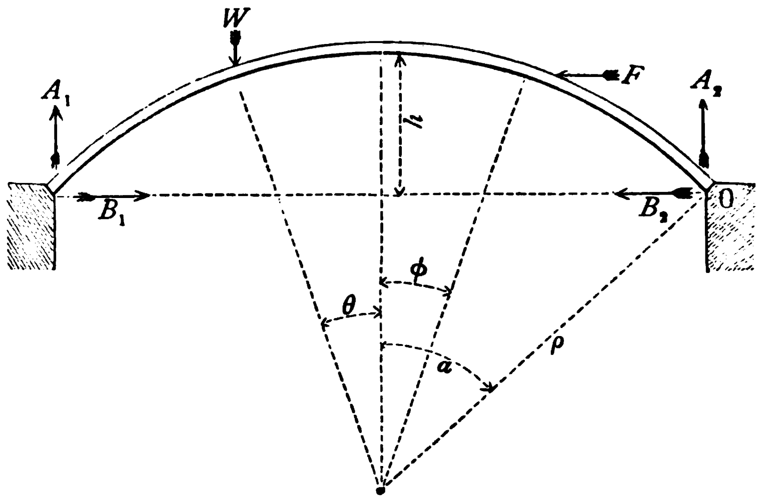


FIGURE 6.

Let O be the origin at one support (see figure);

l the span; h the rise; ρ the radius of curve;

α the angle subtended at the centre by the half-span;

r the radius of gyration of the section $=\left(\frac{I}{S}\right)^{\frac{1}{2}}$;

a the axial distance of the extreme lamina.

The approximate value of B_1, which in this case is equal to B_2, is

$$B_1=\frac{Wh^2}{a^2}\cdot\left\{\frac{4a^2.\alpha^2-(\pi r\sin\alpha)^2}{4a^2h^2+15r^2l^2}\right\}\cdot\frac{G}{K};$$

where $G=\frac{1}{2}(\sin^2\alpha-\sin^2\theta)+\cos\alpha(\cos\theta+\theta\sin\theta)-\cos\alpha-\alpha\sin\alpha$;

$K=\alpha+2\alpha\cos^2\alpha-3\sin\alpha\cos\alpha$.

(2.) When the force is a single representative horizontal force F, its position being given by ϕ the inclination to verticality of the section of its action.

Then $A_1 = -A_2 = F\dfrac{\cos\theta - \cos\alpha}{2\sin\alpha}$;

and if B_3 represent the horizontal reaction at the abutment when F is counterbalanced by an equal and opposite corresponding force in the other half-span,

$$B_1 = \tfrac{1}{2}(B_3 + F);\ B_2 = \tfrac{1}{2}(B_3 - F);$$
$$B_3 = -2F.\frac{l^2.G' + 2r^2\sin^2\alpha(\phi + \sin\phi\cos\phi)}{l^2K + 4r^2\sin^2\alpha(\alpha + \sin\alpha\cos\alpha)};$$

where $G' = \tfrac{1}{2}\phi - \tfrac{1}{2}\sin\phi\cos\phi - \sin\phi\cos\alpha + \phi\cos\phi\cos\alpha$;
$K = \alpha + 2\alpha\cos^2\alpha - 3\sin\alpha\cos\alpha.$

(3.) When the force is simply the stress due to change of temperature, without any load; if τ be the expansion of material per unit of length, and B_4 be the simple horizontal stress resulting.

With a moderately flat curve, $B_4 = \dfrac{15EI.\tau}{15r^2 + 8h^2}$.

Third Case.—Rib of circular curvature and uniform section resting on free supports, having an evenly distributed uniform load.

(1.) Uniform load w per unit of length of arc, this also applying to inherent weight of rib.

If s=length of arc of the whole rib;
l, length of span; h, the rise; ρ, radius of circle;
r, radius of gyration of section $= \left(\dfrac{I}{S}\right)^{\frac{1}{2}}$.

Then $A_1 = A_2 = \tfrac{1}{2}ws$; and with rather flat curvature

$$B_1 = \frac{2wh.\rho.\alpha(7l^2 - 12h^2)}{7l(8h^2 + 15r^2)}.$$

(2.) Uniform load w per unit of length of span, this also applying to the weight of a horizontal platform, or to that of a travelling load covering the whole platform.

Then $A_1 = A_2 = \frac{1}{2}wl$;

$$B_1 = \frac{wh(7l^2 - h^2)}{7(8h^2 + 15r^2)}.$$

Fourth Case.—Parabolic rib of any uniform section resting on free supports, with an evenly distributed load.

Load w per unit of length of span.

Then $A_1 = A_2 = \frac{1}{2}wl$; $B_1 = \frac{wl^2}{8h}$.

Fifth Case.—With partial loads, the values of B_1 are complicated.

If, however, the complete stresses H and T have been already determined for a uniform load w; and if the partial load w_1 extend over half the rib, the values of H and of T due to the combined loads may be obtained from the former by multiplying them by the ratio $\frac{w}{w + w_1}$; similarly also with some other terms involving w. Thus in most such cases the use of a value of B_1 expressly suited to and directly determined for partial loading only is unnecessary.

Solution Number 7.—Compound curved ribs. Strains on the spandrel-bars and on the booms.

When a curved rib is compound, its strength is partly dependent on its upper member and the spandrel-bars between the two members. The treatment to be followed necessarily much resembles that usual with a braced girder of the same general form, for the structural distinction is comparatively small; the spandrel-bars are fixed, while bracing is articulated or free, and the remaining distinctions are matters of dimension.

Before proceeding to the booms and bars, the general equations of equilibrium for the composite rib shown in the figure will be given.

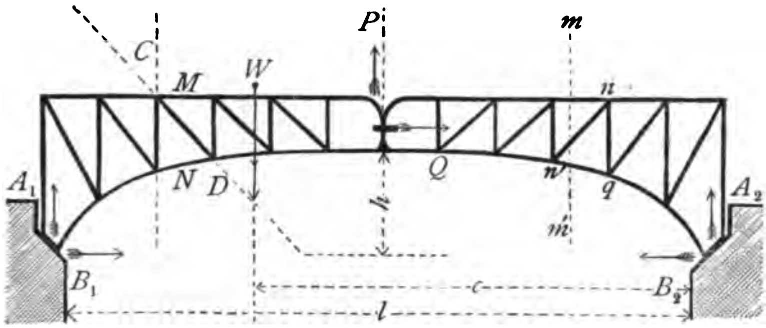


FIGURE 7.

(1) Let W be a single representative weight or load,
ΣW, the sum of all such weights applied,

then for the whole rib,

$A_1 + A_2 = \Sigma W$; $B_1 = B_2$; $Al = c \Sigma W$;

for the half-rib about the middle of the crown,

$Bh = \frac{1}{2}l . A - \Sigma' W . (c - \frac{1}{2}l)$;

where $\Sigma' W$ is the sum of weights on the half-rib.

When the two halves of the rib are unsymmetrical either in form, weight, or load, the reactions P and Q at the crown are

$P = A_2 - \Sigma' W$; and $Q = B_2$;

but with perfect symmetry $P = 0$.

(2) Separate treatment of the bar-stresses.

The stresses on each boom or piece both of the curved and of the horizontal member may be reduced separately, if the whole structure be in perfect accordance with constructive design. But if needless bars exist, such a solution may be indeterminate.

Exemplifying the method on a single boom N of the lower member, let W represent the weight of both rib and load from the left abutment as far as a vertical axis drawn through the left end of the boom N,

R the resistance offered by the boom N,
A the vertical reaction at the left abutment,
B the horizontal reaction at the left abutment,

and w, r, a, b, the respective leverages of those forces with respect to the point C on the vertical axis before mentioned; these leverages are at right angles to the directions of the forces.

Taking moments about the point C,

$$Rr = Aa - Bb - Ww;$$

and R may be either tensile or compressive, as the case will show.

Similarly in the upper or horizontal member, let M be any single boom whose left end is a point C. Taking an inclined axis through CD, the resistance afforded by M prevents the turning of that portion of the rib and its load from the left abutment as far as the axis CD.

Let W' be the weight and load of this portion;
R', the resistance afforded by the boom M;
A and B, the abutment reactions as before;
w', r', a', b', the leverages of these forces with respect to the point D.

Taking moments about the point D,

$$R'r' = Aa' - Bb' - W'w';$$

R' being either tensile or compressive, as the case will show.

The stress on each separate boom from the abutment as far as the middle can be thus separately obtained.

Combining the stresses in pairs at each point of meeting the bars, the stresses along the spandrel-bars are obtained through simple resolution of force.

The method just described is commonly adopted on account of its being apparently easy, but it is not advocated nor is it strictly correct when applied to fixed bars.

(3) Treatment of bar-stresses in the gross.

This method depends on the general shearing stresses throughout the rib. Taking now the right half of the rib for illustration (see figure 7), let the stress on the spandrel-bar nn' be required, and taking any vertical section mm',

let V_1 be vertical component of required stress on nn',

V_2 the vertical component of known stress on boom nq,

ΣW the sum of all weight between mm' and the right-hand abutment,

then $V_1 = A_2 - V_2 - \Sigma W$.

Similarly the shearing stress at any point or through any bar may be successively obtained throughout the rib; employing the method of graphic record to prevent gross error.

In both of these modes of treatment, it has been assumed that V and H have been previously obtained through some general method.

But the values of H, the horizontal stress or bending moment, may be taken approximately for these purposes if required.

Let w be the load, &c., evenly distributed per unit of length of span, over the whole;

d the depth of rib at the crown;

H the horizontal stress at the crown;

then $H = \frac{wl^2}{8h}$ approximately.

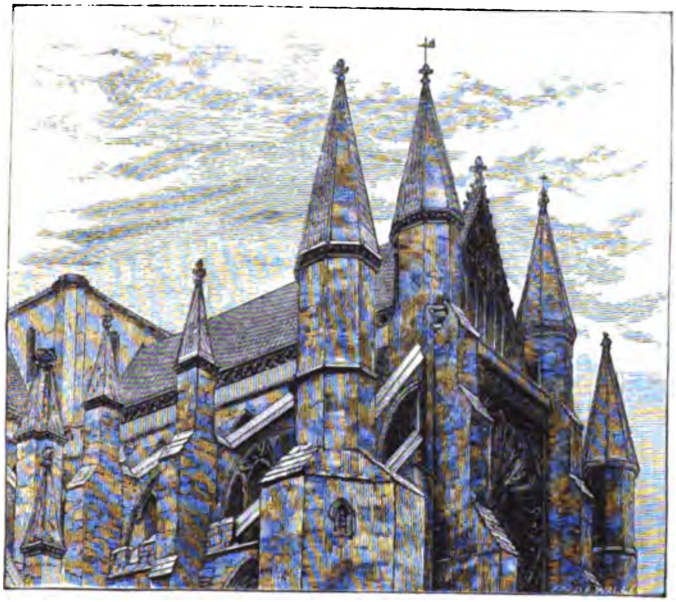
If w extend over only half of the rib,

H_x the horizontal stress at x in the unloaded part,

then in the upper member $H_x = \frac{wl^2}{16h}\left\{\frac{hx(2l-4x)}{dl^2+4hx^2}\right\}$;

and in the lower member $H_x = \frac{wl^2}{16h}\left\{\frac{dl^2+2lhx}{dl^2+4hx^2}\right\}$.

(4) The correct determination of the stresses on the fixed spandrel-bars of an elastic compound curved rib would require a lengthy treatment through elastic deformation, and would vary according to the disposal of the bars in such a way as to render a general solution impracticable.



SECTION III.

ARCHES, ABUTMENTS, WALLS, AND PIERS OF MASONRY OR BRICKWORK.

Number 1.—*Equation to the line of resistance in an arch of circular curvature with a collected load.*

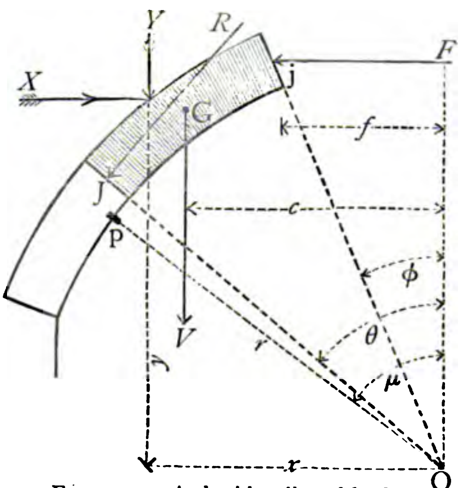


FIGURE 1.—Arch with collected load.

Let Jj be any portion of an arch of circular curvature, having the centre at O; let X and Y be the horizontal and vertical components of the weight of the collected load; F the force acting on the joint j; V the weight of the mass of arch Jj

acting at its centre of gravity, G ; R the resultant of these pressures acting at J.

Let the angle FOJ be θ, angle FOj$=\phi$; let the distance Fj be f, the distance between the verticals FO and GV be c, and the distance OJ be r; let x and y be the co-ordinates of the load from an origin O.

Then by the principle of parallel application of equilibrated forces at J, and by equality of moments about O,

$$(V+Y)r\sin\theta+(F-X)r\cos\theta=Vc+Yx-Xy+Ff.$$

$$r=\frac{Vc+Yx-Xy+Ff.}{(V+Y)\sin\theta+(F-X)\cos\theta}\quad . \quad . \quad . \quad \text{(Eq. I.)}$$

in which V and c are given functions of θ.

Let $r_1\,r_2$ be the radii of the intrados and extrados respectively, for blocks of equal depth,

$$\text{then}\quad Vc={}_{r_1}^{r_2}\!\int{}_{\phi}^{\theta}\!\int r^2\sin\theta.\,\delta\theta.\,\delta r$$
$$=-\tfrac{1}{3}(r_2^3-r_1^3)(\cos\theta-\cos\phi)$$

$$\text{and as}\quad V=\tfrac{1}{2}(r_2^2-r_1^2)(\theta-\phi)$$

$$\therefore\; r=\{\tfrac{1}{2}(r_2^2-r_1^2)(\theta-\phi)\sin\theta+Y\sin\theta-X\cos\theta+F\cos\theta\}$$

$$\text{or}\qquad r=\tfrac{1}{3}(r_2^3-r_1^3)(\cos\phi-\cos\theta)+Yx-Xy+Ff.\quad \text{(Eq. II.)}$$

A polar equation of general application, whence F, the force due to the other half-arch, may be calculated, as all the remaining terms are given quantities.

If the load be distributed instead of collected, the terms substituted for X and Y will be functions of θ and ϕ, and will replace them. This equation can be adapted also to various forms of arch composed of circular curves; thus, for a curve continuous at the crown, the joint j may be made to fall on OF, so then $\phi=0$; also for an equilateral pointed arch, $\phi=30°$, and so on.

The points of rupture.—At the point of rupture p, the arc Jj becomes Jp, the angle θ becomes μ, and as the line

of resistance touches the soffit, $r=r_1$; substituting these values in the general equation, we obtain

$$r_1\{\tfrac{1}{2}(r_2^2-r_1^2)(\mu-\theta)\sin\mu+Y\sin\mu-X\sin\mu+F\cos\mu=\tfrac{1}{3}(r_2^3-r_1^3)(\cos\phi-\cos\mu)+Yx-Xy+Ff \quad . \quad . \quad \text{(Eq. III.)}$$

Or, if we suppose the general equation to take the form $u=0$, where u is a function of F, r, and θ; then $\delta_F u.\delta_\theta F+\delta_\theta u=0$; and assuming $\delta_\theta F=0$, then $\delta_\theta u$ will $=0$.

Applying then this process to the general equation, differentiating it with respect to F and to θ, and putting $\delta_\theta F=0$, when r and θ have been replaced by r_1 and μ; and putting $r_2=r_1(1+b)$ we obtain

$$\frac{Y}{r_1^2}\left\{\frac{f\cos\mu+x\sin\mu}{r_1}-1\right\}=(\tfrac{1}{2}b^2-b)(1-\frac{f}{r_1}\cos\mu)(\mu-\phi)+\frac{f}{r_1}(\tfrac{1}{2}b^2+\tfrac{1}{3}b^3)\sin\mu+\{(\tfrac{1}{2}b^2+b)\cos\mu-(b+b^2+\tfrac{1}{3}b^3)\cos\phi\}\sin\mu.$$

Also, if the distance from the soffit to the point j of the action of F be represented by dr_1, then $\frac{f}{r_1}=(1+d)\cos\phi$; and the above becomes

$$\frac{Y}{r_1^2}\left\{\frac{x}{r_1}\sin\mu+(1+d)\cos\phi\cos\mu-1\right\}=(\tfrac{1}{2}b^2+b)\{[1-(1+d)\cos\phi\cos\mu](\mu-\phi)+(\cos\mu-\cos\phi)\sin\mu\}+d(\tfrac{1}{2}b^2+\tfrac{1}{3}b^3).\sin\mu\cos\phi. \quad \text{(Eq. IV.)}$$

An equation yielding the angle μ, and determining the position of p the point of rupture.

This general equation may be adapted to various modes of loading by giving special values to x and to Y; also to various forms of arch composed of circular curves by giving special values to ϕ.

This solution is due to Moseley.

Number 2.—*Equation to the line of resistance for an arch of circular curvature loaded to an inclined load-line.*

Let the surface of the loading be inclined to the horizon at an angle α; let w be the weight of a cubic unit of arch, w' that of a cubic unit of loading, and let $w'=gw$; also let r_2 be the radius of the extrados as in the foregoing case, and let the depth of loading at the highest point (see figure) be $=cr_2$. Use also the values h and z, shown in the figure.

First to find the moment of the load with respect to O, using notation similar to that in the preceding paragraph.

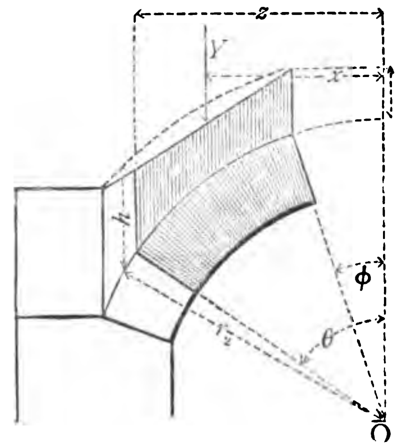


FIGURE 2.—Arch with inclined load-line.

The area of the loading $= \int_{\phi}^{\theta} h_z$;

but $h=r_2+cr_2-(r_2\cos\theta+r_2\sin\theta.\tan\alpha)=r_2\{1+c-\cos(\theta-\alpha).\sec\alpha\}$;

and $z=r_2\sin\theta$;

hence $\int_{\phi}^{\theta} h_z=\int_{\phi}^{\theta} h.\partial_\theta z=r_2^2\int_{\phi}^{\theta}\{1+c-\cos(\theta-\alpha)\sec\alpha\}.\cos\theta.\partial\theta.$

Hence the weight of loading, represented by Y as before,

$$=gr_2^2\int_{\theta}^{\theta}\{1+c-\sec\alpha.\cos(\theta-\alpha)\}\cos\theta=gr_2^2\{(1+c)(\sin\theta-\sin\phi)-\tfrac{1}{4}\sec\alpha[\sin(2\theta-\alpha)-\sin(2\phi-\alpha)]-\tfrac{1}{2}(\theta-\phi)\} \quad \text{(Eq. I.)}$$

Also the moment of the load $= Yx$,

$$=gr_2^3\int_{\phi}^{\theta}\{(1+c)-\sec\alpha\cos(\theta-\alpha)\}\sin\theta.\cos\theta=gr_2^3\{\tfrac{1}{2}(1+c)(\cos^2\phi-\cos^2\theta)-\tfrac{1}{3}(\cos^3\phi-\cos^3\theta)-\tfrac{1}{3}\tan\alpha(\sin^3\theta-\sin^3\phi) \quad \text{(Eq II.)}$$

These values of Y and of Yx given in Eq. I. and II. can be substituted in the general equations given in the preceding paragraph, and thus the required equation to the line of resistance suitable to this will be obtained.

Number 3.—A segmental arch with the upper surface of the loading horizontal.

Treating this as a special case, here $\phi=0$, $g=1$, and $a=0$; also as before, $r_2=r_1(1+b)$; $r=dr_1$; the depth of arch-ring $=br_1$; the depth of loading at the crown $=cr_2$.

Taking the values of Y and Yx given in the last paragraph, reducing their terms to suit these special particulars, and substituting them in Eq. III. the paragraph before (No. 1); the result may be solved with regard to $\frac{F}{r^2}$.

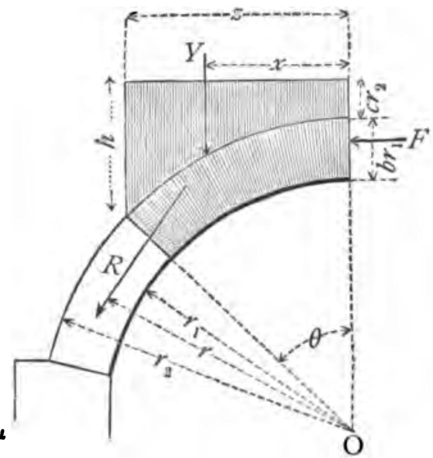


FIGURE 3.—Segmental arch, with horizontal load-line.

Reducing it we obtain

$$\frac{F}{r^2}=\frac{1}{1+d-\cos\mu}\cdot\tfrac{1}{2}(1-b)(1-b)^2.(1+c).\sin^2\mu+\tfrac{1}{6}(1+b)^2(1-2b)\cos^3\mu$$
$$+(\tfrac{1}{2}b^2+\tfrac{1}{3}b^3-\tfrac{1}{2})\cos\mu+\tfrac{1}{2}\mu.\sin\mu+\tfrac{1}{3}\qquad\text{(Eq. I.)}$$

Assuming $\delta F=0$ as before, and putting $d=b$, so that F acts at the upper edge of the keystone, and reducing, the above becomes

$$\tfrac{2}{3}(1-2b)\cos^3\mu-\{(1-b)(1+c)+(1+b)(1-2b)\}\cos^2\mu+\left\{\frac{1}{(1+b)^2}+2(1-b^2)(1+c)\right\}\cos\mu$$
$$+\frac{1}{(1+b)^2}\left\{1-(1+b)\cos\mu\right\}\frac{\mu}{\sin\mu}-(1-b)(1+c)-\frac{2}{3(1+b)^2}\quad\frac{b^2+\frac{2}{3}b^3}{1+b}=0.$$

Also, if $d=0$, so that F acts at the lower edge of the keystone, and assuming $\delta_\mu F=0$, then Equation (I.) becomes

$$\tfrac{2}{3}(1+b)^2(1-2b)\cos^2\mu+(1+b)^2\{(1-b)c+\tfrac{1}{3}(4-5b)\}\cos\mu+\frac{\mu}{\sin\mu}-\{(1+b)^2(1+b)(1+c)+\tfrac{2}{3}+b^2(1+\tfrac{2}{3}b)\}=0.$$

A similar mode may be applied to the pointed arch with inclined loading, but the reduction is tedious and the result very lengthy.

From these formulæ tables have been calculated that give the thrust F, the values of $\frac{F}{r_2}$, and of μ the angle for finding the point of rupture, corresponding to various values of b and of c for segmental arches, loaded to various inclinations.

Having thus the values of $\frac{F}{r_2}$ above given, and working out values of Y and of Yx from Equations I. and II. of the last paragraph (Number 2) they may be substituted in the general polar equation II. of paragraph Number 1; thus the equation to the line of resistance is determined, and the direction of the resultant pressure at any point can be obtained. The two components of this resultant are the weight of the half-arch and the horizontal thrust at the keystone. Hence the magnitude of the resultant may be got for any point down to the abutment, through the general equation afforded by Moseley in paragraph Number 1.

For all ordinary purposes, however, the rougher simple mode of paragraph Number 5, after modification to horizontal load line, is to be preferred.

Number 4.—*The conditions of the arch of circular curvature not less than a quadrant having a horizontal platform above it (alternative method).*

Let $r_1\, r_2$ be the radii of the intrados and extrados respectively,

c the depth of loading above the crown.

$w\; w'$ the weights of a cubic foot of the arch and of the loading including voids respectively.

Then to obtain the angle (μ) for the point of rupture

$$\left\{\frac{w'}{w}\left(1-\cos\mu\right)+\left(1-\frac{w'}{w}\right)\frac{\mu-\cos\mu.\sin\mu}{2\sin^3\mu}\right\}r_2^2+\frac{w'}{w}cr_2-\frac{\mu-\cos\mu\sin\mu}{2\sin^3\mu}r_1^2=0 \quad . \; . \; . \; . \; . \; . \quad \text{(Eq. I.)}$$

This quadratic equation for obtaining μ from given values of r_1, r_2, c, w, and w' can be solved by repeated approximation, after assuming and interpolating certain values of μ.

To obtain the horizontal thrust (H) for each unit of breadth of arch,

$$H=w'r_2^2\left\{\left(1+\frac{c}{r_2}\right)\cos\mu-\tfrac{1}{2}\cos^2\mu-\tfrac{1}{2}\mu\cot\mu\right\}+w(r_2^2-r_1^2).\tfrac{1}{2}\mu\cot\mu. \quad . \; . \; . \; . \; . \; . \quad \text{(Eq. II.)}$$

Next, to find the radius of the extrados r_2 when r_1, c, and μ are given.

Putting (Eq. I.) into the form $Ar_2^2+\frac{w'}{w}cr_2-Br_1^2=0$; we may obtain

$$r_2=\left\{\frac{B}{A}r^2+\frac{w'^2c^2}{4w^2A^2}\right\}^{\frac{1}{2}}-\frac{w'}{2wA} \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(Eq. III.)}$$

in which the values of A and B can be substituted.

Treating the horizontal thrust as nearly equal to the weight supported between the crown and that part of the soffit whose inclination is 45°, and expressing this in symbols, the height of the extrados at the crown above the level of the joint at 45° is equal to $0{\cdot}7071 r_2$; and the thrust is

$$H = w' r_2 (0{\cdot}0644 r_2 + 0{\cdot}7071 c) + 0{\cdot}3927 w (r_2^2 - r_1^2) \quad . \quad \text{(Eq. IV.)}$$

The above are the four equations employed by Rankine.

Number 5.—*Rough modes of ascertaining the horizontal thrust, the total thrust, the points of rupture, and depth of keystone; and of testing the stability by diagram.*

The point of rupture.—Take a point p in the arch somewhere near the probable point of rupture; let the inclination to the horizon of the joint at p be a. Take any

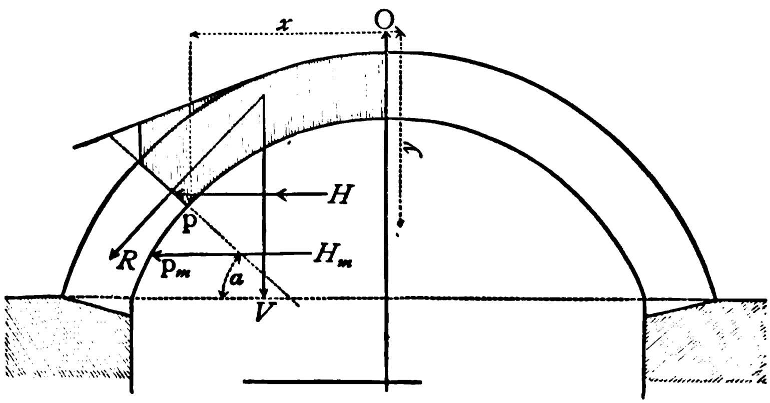


FIGURE 4.—Horizontal thrust of an arch.

point O in the vertical axis of form of the arch passing through the crown as an origin, and let x and y be the co-ordinates of p. Let H be the horizontal thrust at p, R the whole thrust at p, V the vertical load on the whole arc from the crown down to p.

Then $H = V\partial_y x = V.\text{cotan}\,\alpha$;
and $R = (V^2 + H^2)^{\frac{1}{2}} = V.\text{cosec}\,\alpha.$

Compute in this manner values H_1 H_2 H_3, &c., for other points p_1 p_2 p_3, &c., on either side of p, and arrive by interpolation at the highest possible value H_m at p_m; then p_m will be the point of rupture, provided rupture is possible. This method may be used either at the intrados or the extrados of any arch, anywhere but at the crown.

The horizontal thrust at the crown of an equilibrated arch of circular curvature is the weight of the half-arch and its load, multiplied by the half-span, and divided by the rise.

The total thrust.—As the value of R, the whole thrust or resultant pressure at any other joint, may be obtained by the formula already given; and as, even if probable points of rupture be not required, it may be necessary to find the total thrust at some theoretical joint beyond which the remainder of the arch may be treated as abutment; let the thrust R at such a joint J in the figure be found; and let the joint be inclined to the vertical at an angle β.

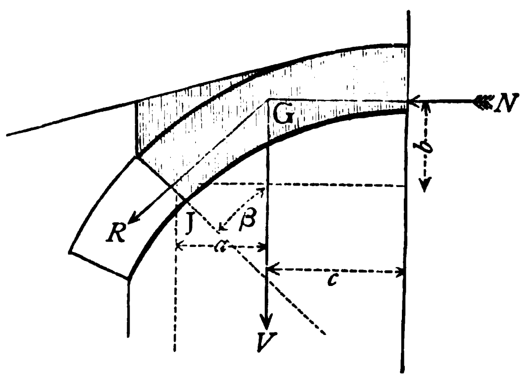


FIGURE 5.—Thrust at the crown of an arch.

Now find the centre of gravity G of the mass of arch and loading down to that joint J, and let V be the weight of that portion acting at G. Let N be the required thrust at the crown, acting at the most unfavourable part of the

central third, in this case the highest; then taking moments about the point J,

$$N=\frac{Va}{b};\ \text{or}\ N=V\tan\beta,\ \text{or}\ N=\sqrt{R^2-V^2}.$$

The following is a method of obtaining V and the value of c, the distance of its action from the axis BD:

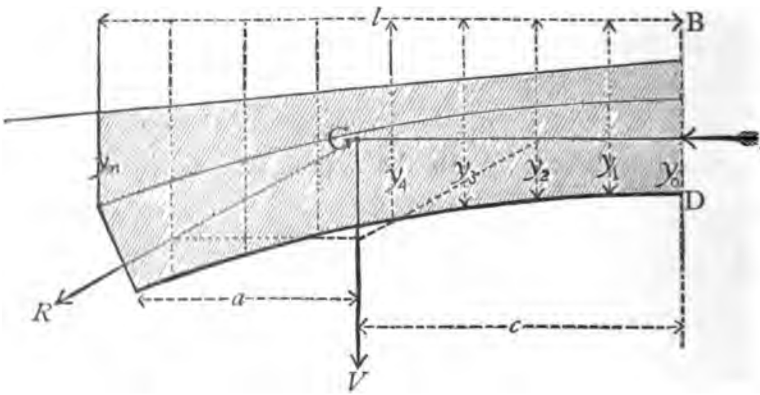


FIGURE 6.—Position of centre of gravity.

Let $V=Aw$; w=weight of a unit of arch and load,
A=the shaded area in figure representing it,
l=the horizontal length of that area,
n=the number of equal parts composing l.

$$\text{Then } c=\frac{\Sigma(ny)}{A}$$

$$=\frac{l}{3n}\left\{y_0 l+4\left(y_1\frac{l}{n}+y_3\frac{3l}{n}+\&c.\right)+2\left(y_2\frac{2l}{n}+y_4\frac{4l}{n}+\&c.\right)\right\}.$$

Then if by diagram any line on GV be made to represent by scale the amount of V, the amount of N and of R can be obtained by scale on the parallelogram of forces, after the whole has been tested and proved to satisfy the conditions of stability.

Correspondingly also a graphic method may be employed

for testing the stability of a proposed arch drawn on paper to a sufficiently large scale. Having found the joint of rupture, mark the centre of gravity of the combined arch and load from the crown down to that joint, and draw a vertical line through the centre of gravity. From the highest and lowest points of the central third of the arch-thickness at the crown, draw two lines parallel to the tangent at the crown. From the lowest and highest points of the central third of the arch-ring at the joint of rupture, draw two lines parallel to the tangent to the intrados. The quadrilateral formed by the intersection of these four lines should fall on the vertical line drawn through the centre of gravity. Should it not do so sufficiently well, work back from any such convenient point on the vertical line to the unfavourable edges of the central thirds at the crown and at the joint of rupture by lines parallel to the before-mentioned tangents; then, if such lines can be made to fall within the assigned limits, the arrangement of load, form of curve, and arch-thickness combine to effect stability; but if not, some one of these three conditions must be modified.

The depth of keystone.—The value of this quantity cannot be analytically determined in any way. A collection of cases, including the Grosvenor Bridge at Chester of 200 feet span, yielded the following empirical expressions of Rankine:

In a single arch $d_1 = \sqrt{0{\cdot}12 R_1}$

In an arch of a series $d_2 = \sqrt{0{\cdot}17 R_2},$

where R_1, R_2 are the radii of curvature of the intrados at the crown in either case, and d_1, d_2 are the corresponding depths of keystone.

As there is not yet any mode of discovering with sufficient exactitude by solution the effect of moving loads on

the lines of pressure and resistance in an arch, an excess of depth of keystone, beyond that necessary to resist crushing under ordinary dead loads, is allowed in ordinary practice in accordance with the above formulæ.

The more exact modes of dealing with arch-rings involve the use of equations to the line of resistance. The cases already dealt with are those most suited to ordinary engineering and architectural purposes, segmental arches of circular curvature, and pointed arches of composite circular curvature.

Treating arches of other sorts generally, it may be noticed that the complete semi-circle, the semi-ellipse, and semi-cycloid would require infinite loads at the haunches, and that the simple catenary demanding a high rise would be inconvenient for a bridge; but the parabola and the

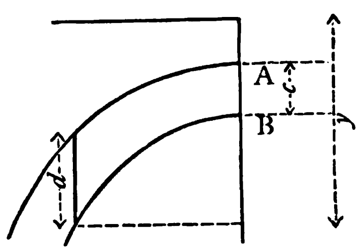


FIGURE 7.—Hyperbolic.

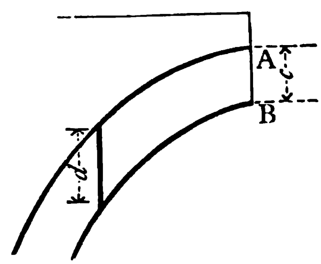


FIGURE 8.—Parabolic.

hyperbola being weak at the haunches are capable of being applied under certain conditions of loading. The latter is less convenient of the two in calculations and solutions; for if we take the equation to it for determining the curve of equilibrium, and obtaining the ratio of the vertical depth of the arch-ring at any point to that at the crown, it becomes $d=\frac{cm^3}{y^3}$, where m=semi-transverse axis (see figure); while in the parabola it is simply $d=c$. Although the parabola is hence convenient in solutions, it yet offers less practical

convenience than a circular segment approximating to it within certain limits. With uniform loading per unit of length of span it closely represents the curve of equilibrium, and then is useful.

The inverted transformed catenarian (see Suspension Bridges) suited to loading proportionate to vertical ordinates is the true curve of equilibrium for an arch simply loaded to a horizontal line, as in the figure. The equation for the curve is that of the simple catenary, with the expressions for vertical forces modified by the factor $\frac{c_1}{c_2}$.

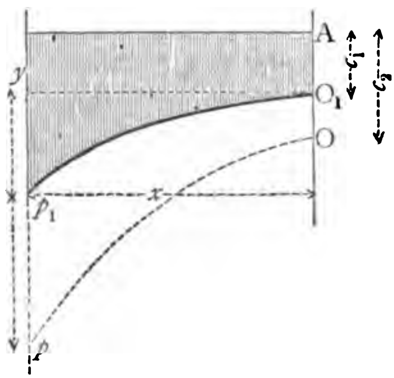


FIGURE 9.—Catenarian inverted.

Number 6.—*The chimney-piece arch, its line of resistance, and the horizontal thrust.*

Treating it first as an ordinary arch, let ABCD be the whole of it, ADjJ any portion of it under consideration as

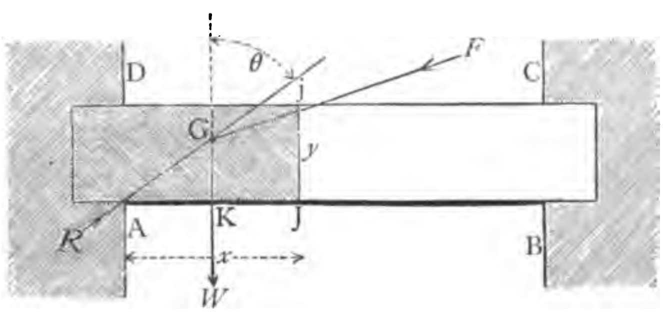


FIGURE 10.— Chimney-piece Arch.

far as vertical point Jj. Let W be its weight acting at its centre of gravity G, let F be the force acting in the direction of jG, that is the resultant pressure of the remainder of the

arch, and let R be the resistance of the abutment acting at A in the direction AG. Then with equilibrium these three directions will meet in G.

Taking A as the origin, let x and y be the co-ordinates of the point j; let $AB=2l$, $BC=h$; let R act at an angle θ with the vertical, and let w=equal the weight of a cubic foot of the arch.

Equating the moments about j,

$$R(x\cos\theta-y\sin\theta)=W.\tfrac{1}{2}x=\tfrac{1}{2}hwx^2;$$

resolving R vertically, we have $R\cos\theta=wlh$;

hence $l(x-y\tan\theta)=\frac{1}{2}x^2$; . . . . (Eq. I.)

a parabolic equation for the lines of resistance.

Examining the conditions shown we find that when $x=l$, and y becomes y', then $y'\tan\theta=\frac{1}{2}l$, and θ decreases as y' increases; but as R decreases with θ, R decreases also as y' increases. Now the maximum value of y' is the arch-depth; hence when $y'=h$, R has its minimum value. That is, when the line of resistance touches the extrados, the value of R compatible with safety is the actual resistance of the abutment at A.

In that case the equation $y'\tan\theta=\frac{1}{2}l$ becomes $\tan\theta=\frac{1}{2}\frac{l}{h}$. Eliminating θ between this and one of the foregoing equations, we get $R=wlh\left\{1+\frac{l^2}{4h^2}\right\}^{\frac{1}{2}}$;

and comparing this with the other preceding equation, we get $R\sin\theta=\frac{1}{2}wl^2$ . . . . (Eq. II.)

and this is the value of the horizontal thrust at A, and is independent of the arch-depth h.

This being so, any uniform vertical loading may be treated as increased depth of arch, and will not diminish the stability as regards position. The stability of the abut-

ments is also independent of the forms of blocks composing the arch.

Frictional stability.—To prevent any block from slipping on any other, the greatest inclination of R to the horizon must be less than the limiting angle of resistance of friction ϕ; that is, when $90^\circ - \alpha < \phi$.

But from the preceding investigation we have

$$\tan\alpha = \tfrac{1}{2}\frac{l}{h}, \text{ or } \cot\alpha = \frac{2h}{l};$$

the condition hence is that $2h < l\tan\phi$.

The liability to slip hence is less with a deep arch. The blocks may be built with radiated joints as an additional protection against sliding, but this does not affect the abutments or the amount of pressure on them.

Number 7.—The Powder-magazine Arch.

Let β be the required inclination to the vertical of the joint of rupture p, and α the given inclination of the sloping exterior surface;

let W be the weight of the whole down to p,

N the horizontal thrust at the extrados of the crown,

a and b the leverages of W and N respectively,

R and r the exterior and interior radii.

Then with equilibrium

$Wa = Nb$.

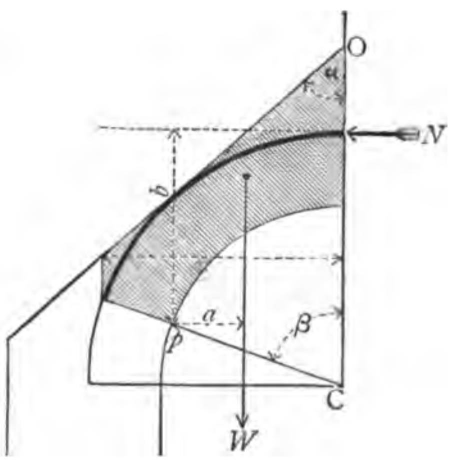


FIGURE 11.—Powder-magazine Arch.

But Wa = moment of trapezoid − moment of sector,

$$= M_1 - M_2 = Tt - Ss.$$

The value of T the area of the trapezoid may be reduced to

$$\tfrac{1}{2}R^2.\sin\beta.\operatorname{cosec}\alpha\,[2-\sin(\alpha+\beta)],$$

and the horizontal distance t of the centre of gravity of T from the point p is equal to

$$r\sin\beta-\tfrac{1}{3}R\sin\beta\left\{\frac{3-2.\sin(\alpha+\beta)}{2-\sin(\alpha+\beta)}\right\}.$$

Also the value of S the area of the sector may be reduced to $\frac{1}{2}\beta r^2$, and its leverage s to

$$r\sin\beta-\tfrac{1}{4}r.\frac{1}{\beta}.\sin^2\tfrac{1}{2}\beta.$$

Hence the moment of the sector M_2 is

$$=r^3\sin\tfrac{1}{2}\beta\{\beta.\cos\tfrac{1}{2}\beta-\tfrac{2}{3}\sin\tfrac{1}{2}\beta\}\text{ ; and}$$

the value of b is $R-r\cos\beta$; whence

$$N=\frac{Wa}{b}=\frac{M_1-M_2}{R-r\cos\beta}=\frac{Tt-M_2}{R-r\cos\beta}.$$

Substituting in this the values of T, t, and M_2, N can be obtained for any assumed values of β. The results of this formula have been tabulated by Petit.

Number 8.—*The horizontal thrust of a Dome.*

Let AD represent the section of half the dome, let AC$=\frac{1}{2}l$, the half-span ; DC the height of the dome. Draw AB tangential at the springing, and BC=h.

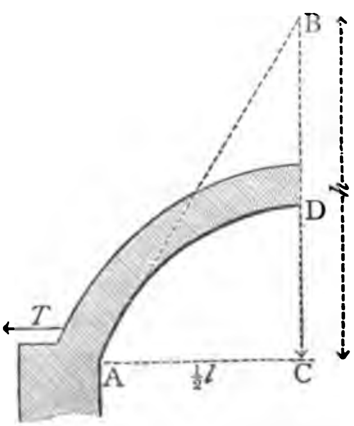


FIGURE 12.—Thrust of Dome.

Then if T=the whole thrust over the circumference at the base of the dome, and V= the whole weight of the dome,

$$T=\frac{l.V}{2h}.$$

In domes of spherical section, the limiting largest segment that can be safely employed is 102°.

As in domes there is both

horizontal and vertical bond, perforations may be made anywhere in it without prejudice to safety. The result at the base may be partially or entirely counteracted by a ring of hoop-iron.

ABUTMENTS, PIERS, WALLS, ETC.

Number 1.—*Abutment or Wall of rectangular section subjected to lateral pressure.*

If the abutment be composed of a number of horizontal courses, as of brickwork or masonry, the curve of thrusts may be determined graphically as before explained in Part I., pages 41, 42, and 48 to 50.

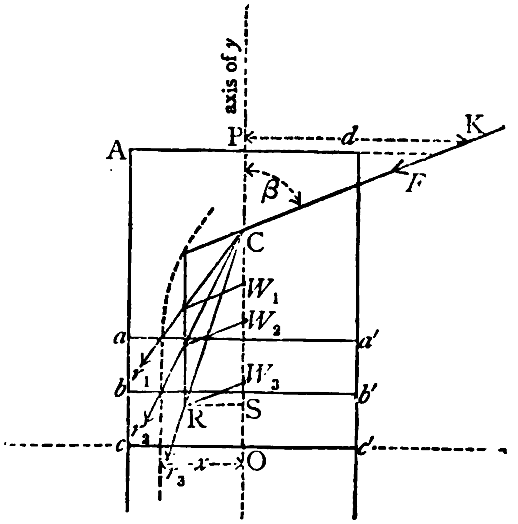


FIGURE 13.—Abutment of rectangular section.

Curve of Thrusts.—To determine analytically the equation to the curve of thrusts, let CR be one of the diagonals (in the figure); draw RS and PK parallel to cc'.

Then let w=weight of a cubic foot of the wall;
let b=the breadth of the wall;
let PK$=d$, and the angle PCK$=\beta$;
also let OP$=y$, and O$r_3=x=\frac{1}{2}b-nb$.

Then in the similar triangles COr_3, RSW$_3$,

$$\frac{x}{y-d\cot g\beta}=\frac{F\sin\alpha}{bwy+F\cos\beta},$$

$$\text{or } x=\frac{F.(y\sin\beta-d\cos\beta)}{bwy+F\cos\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(Eq. I.)}$$

And by transferring the origin and adopting other co-ordinates parallel to these, it will be seen that the line of resistance is a rectangular hyperbola.

Equilibrium requires that this line of resistance shall fall within the abutment down to its foundation; and stability requires that it shall fall within the safe or inner two-thirds of the breadth b; this is otherwise expressed by giving a value of 0·333 to m the modulus of stability. For stability as regards friction, so that the curves or surfaces may not slide on each other, the angle of thrust must be less than the limiting angle of friction for those surfaces.

The greatest height.—To arrive at an expression for the greatest height of abutment consistent with stability, under given data, the values of the co-ordinates x and y will then be

$$y = \mathrm{PO} = \text{the greatest height } H;$$
$$x = \mathrm{O}r_3 = \tfrac{1}{2}b - mb;$$

substituting these values in the general equation (Eq. I.) we obtain

$$H = \frac{F.\cos\beta(b - 2mb + 2d)}{2F.\sin\beta - b^2 w(1 - 2m)} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(Eq. II.)}$$

The least thickness.—Correspondingly also the least breadth of wall (b) at the base that is consistent with the condition of stability may also be determined through (Eq. I.); taking y as the given height, and putting $x = \frac{1}{2}b - mb$, this value of x may be substituted, and the equation solved to find b.

$$b = \left\{\left(\frac{F\cos\beta}{2wy}\right)^2 + \frac{2F}{w - 2mw}\left(\sin\beta - \frac{d}{y}\cos\beta\right)\right\}^{\frac{1}{2}} - \frac{F\cos\beta}{2wy} \quad \text{(Eq. III.)}$$

The effect of varying the distance d of the point of application of F may be also shown by solving the same equation with regard to d.

It may be noticed that the abutment to an arch to be in accordance with the above principles should be built in radiating courses, and not in horizontal courses as a house-wall which supports weight without thrust.

A pier differs from an abutment in that it may have to sustain two pressures, one from each arch on either side of it; it should therefore be sufficiently strong to resist either of these independently, during the period of construction.

Number 2.—The buttressed abutment, when the buttress is of uniform thickness and rectangular section.

In order to reduce the conditions of this case to that of a continuous compound wall, it may be assumed that a single buttress extends throughout the whole length of the wall, while its specific gravity is theoretically reduced to compensate for the extension. The whole will then consist of two parts, one the wall proper with its true specific gravity, the other a theoretical wall at its back having a low theoretical specific gravity. The transformation affects neither the equilibrium nor the stability.

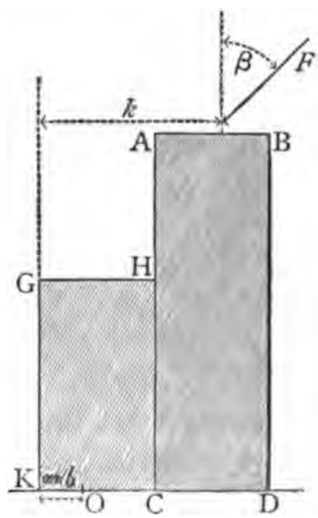


FIGURE 14.—Buttressed wall or abutment.

Let w_1 w_2 be the weights per cubic foot of the wall and the backing after this transformation; let b_1 b_2 be the thicknesses of the wall and the backing; let h_1 h_2 be the heights of the wall and the backing, and mb the distance from the external foot of the backing, within which the line of resistance must not arrive, in order to preserve stability; and let k represent the distance from the vertical axis KG of

the point of application of the force F at AB. (See figure.)

Taking separately the moments of the forces about the point O, they are thus,

Moment of $F = F\{h_1 \sin\beta - (k-m)b\cos\beta\}$

„ W_1; (W_1 (being the total weight of the wall) $= h_1 b_1 w_1 \{\frac{1}{2}b_1 + b_2 - mb\}$

„ W_2; (W_2 being the total weight of the backing) $= h_2 b_2 w_2 \{\frac{1}{2}b_2 - mb\}$.

Equating these moments, the former is equal to the sum of the two latter, hence the general equation

$$F\{h_1 \sin\beta - (k-mb)\cos\beta\} = h_1 b_1 w_1 \{\tfrac{1}{2}b_1 + b_2 - mb_2\} + h_2 b_2 w_2 \{\tfrac{1}{2}b_2 - mb_2\}.$$

By taking known values of w_1, w_2, m and F, and assuming either the value of the ratio of h_2 to h_1 or that of b_1 to b_2, the value of either h_1 or b_1 may be obtained according as one or the other is given or required. (See the method adopted in the last paragraph, Number 1.)

Alternative method.—There is also an approximative mode of arriving at a mean moment of stability for the whole buttressed abutment per unit of length. It consists in taking the moments separately for the simple wall-section and for the buttressed wall-section, and dividing the sum of the two by the distance between the buttresses in plan from middle to middle. The mean moment thus obtained is approximately correct for a mean wall of uniform specific gravity, and the corresponding mean thickness for such a wall is

$$b_3 = \frac{l_1 b_1^2 + l_2 (b_1 + b_2)^2}{l_1 + l_2}$$

where l_1 and l_2 are the lengths in plan of the unbuttressed and of the buttressed portions respectively.

The ratio of the quantity of masonry in the actual buttressed walling to that of the masonry in such a mean wall of the same weight is

$$\frac{l_1 b_1 + l_2(b_1 + b_2)}{b_3(l_1 + l_2)};$$

this ratio being always less than unity.

This approximative method is that adopted by Rankine.

Number 3.—The stepped-buttressed abutment.

Adopting a method analogous to that just adopted in the last paragraph, Number 2; and taking the values of the heights and breadths of the component rectangles as given in the attached figure; let m be the distance from the external foot of the buttress to which the line of resistance is limited in its intersection with the base.

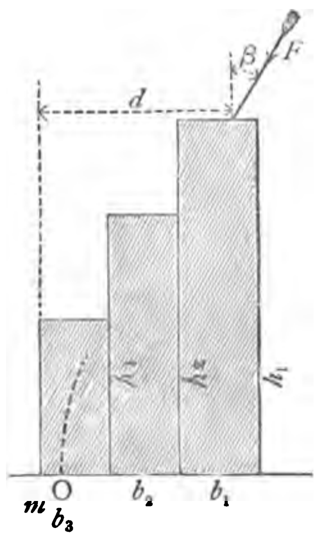


FIGURE 15.—Stepped buttress.

The terms w_2, w_3, being identical as applied to the portions of the stepped buttress, each of these may be put $=\frac{w_1}{r}$, r being the ratio of the reduction of specific gravity for a continuous backing in lieu of detached buttresses.

Equating the moments of the forces about the point O as before, we obtain

$$\begin{aligned} F\{h_1 \sin\beta - (d-m)\cos\beta\} = & \{\tfrac{1}{2}b_1 + b_2 + b_3 - m\} h_1 b_1 w \\ & + \{\tfrac{1}{2}b_2 + b_3 - m\} h_2 b_2 \frac{w}{r} \\ & + \{\tfrac{1}{2}b_3 - m\} h_3 b_3 \frac{w}{r}. \end{aligned}$$

From the above relation between the dimensions and the stability, any one required dimension may be found with reference to a predetermined modulus of stability. The stability should, for economy as well as for balance, be the same at each step of the buttress.

Number 4.—The sloping abutment.

Let $a\,a'$ be any horizontal line in the abutment where the line of resistance is to be found, and let CQ be a vertical axis passing through the centre of gravity C of the portion of section resting on $a\,a'$.

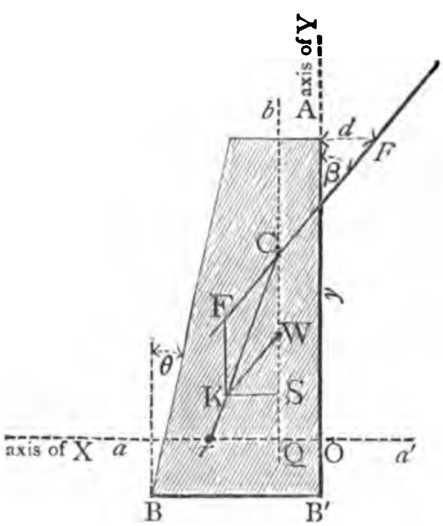


FIGURE 16.—Abutment with sloped back.

Let the line CF represent the force F in direction and magnitude which tends to overturn the abutment; let CW represent the weight W of the section down to aa'; then CKr is the direction of the resultant, and r is a point in the required line of resistance.

Let x be the horizontal co-ordinate of r; $AF=d$;

θ=the inclination to verticality of the back;

β=the inclination of FC to the vertical;

and w=the weight of a cubic foot of the abutment.

Let $AO=y$; let the top breadth$=b$; then $QO=c$ is one of the co-ordinates of the centre of gravity of the section dealt with; and

$$c=\frac{\frac{1}{3}y^2.\tan^2\theta+by\tan\theta+b^2}{y\tan\theta+2b}.$$

Then in the similar triangles $\mathrm{C}r\mathrm{Q}$, CKS we have

$$\frac{r\mathrm{Q}}{\mathrm{CQ}}=\frac{\mathrm{KS}}{\mathrm{CS}}$$

where $r\mathrm{Q}=x-c$; $\mathrm{CQ}=y-(c+d)\operatorname{cotg}\beta$;

$$\mathrm{KS}=\mathrm{KW}.\sin\beta=F\sin\beta;$$

$$\mathrm{CS}=\mathrm{CW}+\mathrm{WS}=\tfrac{1}{2}wy(2b+y\tan\theta)+F\cos\beta.$$

Hence

$$x-c=\frac{F\sin\beta\{y-(c+d)\operatorname{cotg}\beta\}}{\frac{1}{2}wy\{2b+y\tan\theta+F\cos\beta}$$

$$x=\frac{\frac{1}{2}cwy\{2b+y\tan\theta\}+F(y\sin\beta-d\cos\beta)}{\frac{1}{2}wy\{2b+y\tan\theta\}+F\cos\beta}.$$

But from the value of c before given, we may obtain

$$\tfrac{1}{2}cwy(y\tan\theta+2b)=\tfrac{1}{6}wy^3\tan^2\theta+\tfrac{1}{2}wy^2b\tan\theta+\tfrac{1}{2}wyb^2.$$

Hence

$$x=\frac{\frac{1}{3}wy^3\tan^2\theta+wby^2\tan\theta+wyb^2+2F(y\sin\beta-d\cos\beta)}{wy\{2b+y\tan\theta\}+2F\cos\beta}$$

the required equation to the line of resistance.

This equation may be solved to determine the value of any single quantity in accordance with the conditions of stability, in the same manner as was adopted in Paragraph No. 1 for the vertical abutment.

If the slope or batter of the abutment be the quantity specially required, the above equation may be solved with respect to $\tan\theta$, noticing that b the top width and y the height being given, the breadth of base $b+y\tan\theta$ becomes also a quantity involving $\tan\theta$.

If the sloping abutment happen to be buttressed, the principles applied to the buttressed rectangular abutment (Number 2) may be employed to obtain a corresponding equation from which the value of any one required quantity may be deduced.

Number 5.—General conditions of stability of position for an abutment or wall.

Using the following symbols, but retaining y vertical and x horizontal co-ordinates,

W = the weight of the mass,

w = the weight of a unit of the mass,

h, b, t = the height, breadth, and thickness of the structure,

n = a numerical factor dependent on its form and the obliquity of its angles,

q = the safe ratio of deviation from the *middle* of the bed to the thickness of masonry at the given bed-joint,

rt = the distance from the *middle* of the bed to the point where the bed is cut by the vertical axis passing through the centre of gravity,

α = the inclination to the horizon of a line in the plane of the bed,

F = the magnitude of any external force tending to overturn the mass,

θ = inclination to the horizon of the direction of F, contrary to that of α,

y' and x' = the vertical height and horizontal distance of the point of application of F from the centre of resistance of the bed (limited by q).

In accordance with this notation,

$$W = nwhbt;$$

the moment of W with regard to the centre of resistance $= W(q \pm r)t\cos\alpha = n(q \pm r)whbt^2\cos\alpha$;

the moment of F with regard to the same point is

$$F(y'\cos\theta - x'\sin\theta).$$

Hence the condition of stability of position is that
$F.(y'\cos\theta - x'\sin\theta) < n(q \pm r)whbt^2\cos\alpha$ . . . . . Eq. I.

For stability as regards friction,

let ϕ=angle of repose of the material with the horizon,

β=angle with the vertical made by the resultant pressure;

then $\beta = \tan^{-1}\dfrac{F\cos\theta}{W+F\sin\theta}$,

and the condition of stability of friction is

that $\beta - \alpha < \phi$ . . . . Eq. II.

The least thickness (t) at the bed-joint consistent with stability of position may be deduced from Eq. I. put as an equation, in which $x' = t(q+\frac{1}{2})$, whence

$n(q+r)whbt^2\cos\alpha = F\{y'\cos\theta - (q+\frac{1}{2})t\sin\theta\}$.

To simplify the form of this quadratic, use

$A = \dfrac{Fy'\cos\theta}{n(q+r)whb\cos\alpha}$; $B = \dfrac{F\sin\theta(q+\frac{1}{2})}{2n(q+r)whb\cos\alpha}$,

so that the above equation becomes

$$t^2 = A - 2Bt,$$

whence $t = (A+B^2)^{\frac{1}{2}} - B$ . . . . Eq. III.

To determine the least weight of material above the point of action of F that is consistent with stability of friction.

As the utmost obliquity of pressure occurs at the joint or course immediately below the point of application of the force F, the dimensions and weight of the superincumbent mass must be taken into consideration. Let these be W_1, h_1, b_1, t_1, applying them in Eq. II., and assuming $\alpha = 0$, it becomes at the limit

$\dfrac{F\cos\theta}{nwh_1b_1t_1 + F\sin\theta} = \tan\phi$,

or $W_1 = nwh_1b_1t_1 = F\left(\dfrac{\cos\theta}{\tan\phi} - \sin\theta\right)$ . . . . . . Eq. IV.

This is the method, and these are the equations of Rankine, here introduced for those that prefer them, the only alteration being that in accordance with the system applied in this book the co-ordinates are conversely symbolised.

Number 6.—*The shored wall of rectangular section.*

The general condition of stability of a simple rectangular wall or abutment subject to an external force F acting at an angle ϕ with the vertical, is given in paragraph Number 1; it is

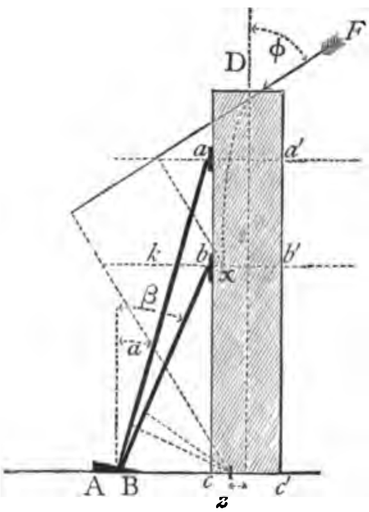


FIGURE 17.—Shored wall.

$$x = \frac{F.(y \sin\phi - d \cos\phi)}{bwy + F \cos\phi};$$

where y = height of the portion of wall under consideration,

b = breadth of the wall,

d = distance from the middle of the top breadth of the wall to the point of action of F,

x = distance from the middle of the base of the wall to the limiting point for the action of the resulting pressure.

If we now apply two shores, Aa, Bb, to such a wall that may require them in order just to attain the requisite stability, the effect will be to leave the top portion of the wall down to the course aa' totally unaffected, and hence remaining under the before-mentioned general condition; while the effect of the shore Aa is to ensure stability down to the course bb', and that of the shore Bb is to prevent the

wall turning over at the base cc' at its centre of resistance there.

Now the point in the bed-joint aa' where the line of resistance cuts it may be obtained through the equation above given; it is x, and may be denoted by $\frac{1}{2}b-m_1b$ in terms of the breadth of the wall, and the distance from the outer foot mb.

Let the corresponding distances of the line of resistance from the outer edge of the wall at the bed-joint bb' and at cc' be denoted by m_2b and m_3b; let the angles made with the vertical by the shores Aa and Bb be respectively α and β; let the thrusts on those shores be respectively T and T_1, the weights of the shores $2v$ and $2v_1$.

Also let $aD=h_1$, $bD=h_2$, $cD=h_3$,

$bk=k$, $cA=k_1$, $cB=k_2$.

Now taking moments about z the point where the line of resistance will cut the base, we get

$$T(k_1+m_3b)\cos\alpha+T_1(k_2+m_3b)\cos\beta+wbh_3(\tfrac{1}{2}b+m_3b)$$
$$=F\{h_3\sin\phi-(d+\tfrac{1}{2}b-m_3)\cos\phi\}+(v+v_1)m_3 \quad . \quad . \quad . \quad \text{Eq. I.}$$

To obtain now the value of T_1 it is necessary to deal with the portion of wall down to the course bb', and treat that as the base; taking moments about x the point in which the line of resistance will cut bb', we get

$$T(k+m_2b)\cos\alpha+wbh_2(\tfrac{1}{2}b-m_2b)=$$
$$=Fh_2\sin\phi-(d+\tfrac{1}{2}b-m_2)\cos\phi+vm_2,$$

this being solved with regard to T becomes

$$T=\frac{1}{(k+m_2b)\cos\alpha}\cdot\Big\{F[h_2\sin\phi-(d+\tfrac{1}{2}b)\cos\phi]-\tfrac{1}{2}wb^2h_2$$
$$+m_2(F\cos\phi+wbh_2+v)\Big\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{Eq. II.}$$

By substituting in Equation I. this value of T, the value

of T_1 may be obtained, as all the remaining terms are known quantities.

If the wall be equally stable at bb' and at cc', then $m_2 = m_3$, and the equation is slightly simplified. When $m_2 = m_3$, and each are $=0$, there is merely simple equilibrium, and the wall is on the verge of destruction.

Number 7.—The house-wall.

When the floor-joists at each story of a house are notched into wall-plates, and these are firmly built into the walls, the former act as ties, and thus help to prevent the walls from falling outward. Even when the joists are not notched but merely rest on the wall-plates, the friction at their ends serves to make the former act as ties as in the other case, but to a less degree.

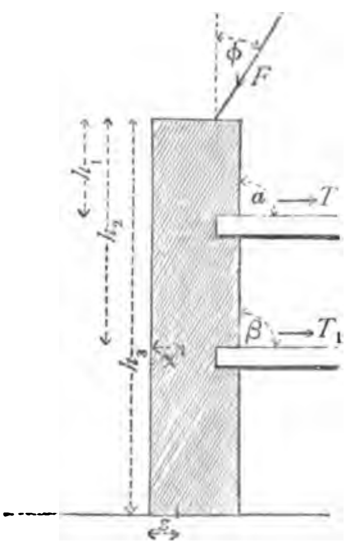


FIGURE 18.—House-wall.

Proceeding on either assumption, the effect of such ties corresponds exactly to the effect of the shores in the shored wall (Number 6).

Hence we may use the equations there given with the following modifications; as the external forces are horizontal, and v and v_1 will be the weight of the floorings resting on the joists.

Put α and β each $=90°$,
for vm_2 and vm_3 put $(b-m_2)v$ and $(b-m_3)v$,
and for $(v+v_1)m_3$ put $v+v_1)\ (b-m_3)$.

The results will enable the values of T and T_1 to be determined, as the equations will hold for conditions of stability in this case.

Number 8.—The abutment of a chimney-piece arch.

The investigation of the chimney-piece arch is given in Paragraph Number 6 on Arches; that of a rectangular abutment is given in Number 1 on Abutments.

Taking from the latter the equation to the line of resistance in an ordinary abutment

$$x=\frac{F.(y\sin\theta-d\cos\theta)}{wby+F\cos\theta}$$

and applying it to the chimney-piece abutment it is evident that the following values given in Number 6 will correspond exactly to $F\sin\theta$ and $F\cos\theta$, and may be substituted for them.

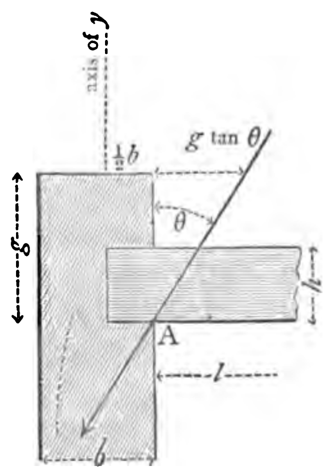


FIGURE 19.—Abutment of chimney-arch.

$F\sin\theta=R\sin\theta=\frac{1}{2}wl^2$
$F\cos\theta=R\cos\theta=wlh.$

Also, if g be the height of the abutment above the point A, the quantity $d=\frac{1}{2}b+g\tan\theta=\frac{1}{2}b\left(1+\frac{l}{h}\right)$; hence by substitution

$$x=\frac{l^2\left\{y-\left(g+\frac{hb}{l}\right\}\right.}{2(by+hl}=\frac{l^2(y-g)-hbl}{2(by+hl},$$

the required equation to the line of resistance;
where b is the breadth of the abutment,

h and l are the depth and length of the arch,

w is the weight per cubic foot of the abutment or of the arch when similar.

Number 9.—The chimney-shaft, tower, or light-house section.

When a shaft or tower is comparatively unloaded, of small section, and has to be carried to a great height, the ordinary section of stability may be first deduced under the supposition that there is not any loading or any unbalanced external pressure or force. The problem under such conditions will determine a section of masonry of such a form that some limiting pressure shall not be exceeded at any point, while the material shall be economically distributed. The practical conditions require the section to be symmetrical about an axis, and demand that the uppermost part of it will be vertical and rectilinear, or at least nearly so when lofty. As the lower part of the section will necessarily have greater breadth, we may assume that this will be bounded by curves convex to the axis, or by rectilinear compound forms approximating to such curves.

First, let us determine the height that the section will bear when kept entirely vertical without exceeding the limiting pressure; the curve for the lower part may be afterwards decided.

Taking the simple section ABCD, with a base CD,

let h be the maximum height of the vertical section,
w the weight of a cubic foot of the masonry,
b the breadth of the base,
R the limiting pressure on the base,
P the vertical force distributed over the base.

The section being symmetrical in form, the vertical axis will pass through its centre of gravity and bisect its base; hence $R = wh$, or $h = \frac{R}{w}$; the greatest pressure at C will be

$=\frac{P}{b}$; the condition of stability being that $\frac{P}{wb}$ will be equal to or less than h; or at the extreme

$$b=\frac{P}{wh} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(Eq. I.)}$$

The pressure per superficial unit never exceeding wl as a necessary condition, we may assume that at a lower point in the section, at an infinitely small depth below the base before fixed for a vertical section, the pressure remains the same, and we may proceed by differentiation to find the curve of the lower part DdK. The increase of surface in the base must be proportional to the increase of pressure in the whole section, and as symmetry exists on either side of a vertical axis, hence we may treat only half the section.

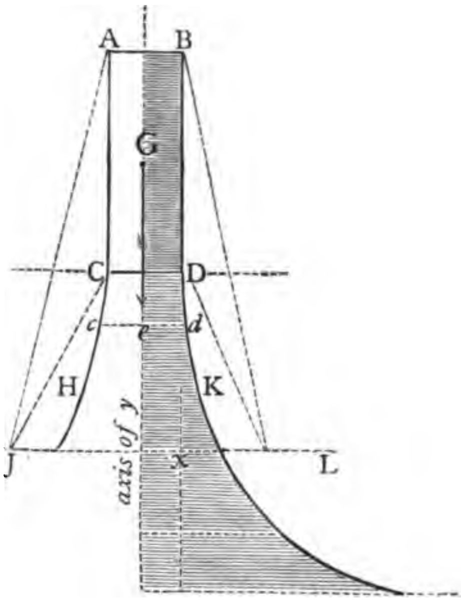


FIGURE 20.—Ultimate position.

Let x and y be the co-ordinates of the curve; but let the values of y be treated as positive when measured downwards from CD towards the axis of x, but negative when upwards from CD;

A = the half sectional area at CD;

P = the pressure from above on half of CD;

$R=wh$, the limiting pressure per unit of surface.

We may consider also that a solid portion of section has

a width of unity, or any constant width in a plane at right angles to it $=t$.

This condition is expressed by $\partial P = R\partial A = Rt\partial x$; also $\partial P = wtx\partial y$;

hence $Rt\partial x = wtx\partial y$; or $wht\partial x = wtx\partial y$; and $\partial y = \frac{h}{x}\partial x$; and

by integration $y_1 - y_2 = h\log\left(\frac{x_1}{x_2}\right)$.

When x_2 is taken $=h$, then $\frac{\partial y_2}{\partial x_2} = 1$, and $y_2 = 0$; that is, if

the origin of the co-ordinates is set at the point where $x_2 = h$; the tangent to the curve will there be inclined to the axis of x at the angle 45°. Hence by substitution

$$y = h\log\frac{x}{h} \quad . \quad . \quad . \quad . \quad . \quad \text{Eq. II.}$$

The curve is therefore logarithmic; and the complete curve, having the axis of y for an asymptote, gives the form for a section of infinite height, when the limiting pressure per unit of surface on any horizontal section is R, or wh.

The above formula is for Naperian logarithms, and with common logarithms becomes

$$y = h \,.\, 2{\cdot}302585 \,\text{Log}\frac{x}{h},$$

whence the breadth of the base $2x$ may be found, remembering that y has a negative sign.

The determination of the logarithmic curve is assigned to Delocre.

Conical Tower.—If inclined straight lines CJ, DL be adopted instead of curves for the lower portions C*c*H, D*d*K. Let a be the top width, $x =$ the breadth of base, and $H =$ the total height of the whole; then the equation for breadth of base becomes

$$\left\{ha+\left(H-h\right)\left(\frac{a+x}{2}\right)\right\}\frac{w}{x}=wh\,;$$

$$\text{or } x=a.\frac{h+H}{3h-H}; \quad . \quad . \quad . \quad . \quad \text{Eq. III.}$$

But if the inclined straight lines, instead of commencing from C and D, commence sloping from the top at A and B, then

$$H\left(\frac{a+x}{2}\right)\frac{w}{x}=wh\,;$$

$$\text{or } x=\frac{Ha}{2h-H} \quad . \quad . \quad . \quad . \quad . \quad \text{Eq. IV.}$$

Hence under these circumstances, when $H=2h$, x is infinite; and when H is greater than $2h$, it will be negative. In other words, when the limiting resistance of the masonry is not exceeded, the maximum height may be as much as twice the maximum height of a section having vertical faces.

To exemplify the effect of Equations III. and IV., let the weight of 1 cubic foot of the masonry be twice that of 1 cubic foot of water, or put $w=2$; let R be 200 talents[1] per square foot; then $h=\frac{R}{w}=100$ feet.

Assuming $H=164$ feet, and top width $a=16{\cdot}4$ feet; then Equation III. yields x for breadth of base $=31{\cdot}83$ feet. But Equation IV. yields x for breadth of base $=74{\cdot}66$ feet, and Equation II. gives a result near that of III.

Finally it may be noticed that any tower, being liable to lateral pressure from wind, on any side of it from top to bottom, might require a section in excess of the above, after allowance by a factor of safety for material. Yet in any

[1] The English talent or foot-weight of water at its utmost density is very nearly 1000 ounces, or 62½lbs. It is the best and most convenient scientific unit of weight.

such extreme case, the actual quality of material and of work would be more important.

Number 10.—*The bridge-pier of solid masonry or brickwork.*

The stresses on a bridge-pier and the strains or resistances offered at every horizontal section by the material have been treated; it now merely remains to mention the general conditions of stability, and the circumstances or sets of external conditions under which the stress and safe strain have to be equated.

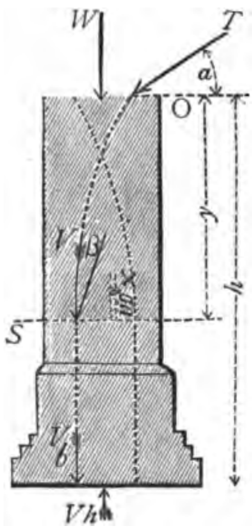


FIGURE 21.

First, the general conditions of stability. To prevent overturning at any horizontal course the curve of thrust must fall within the pier everywhere. Stability requires also that it should fall within the central half.

To prevent tension the curve of thrust should intersect the effective breadth of course everywhere.

Hence, to limit the breadth of pier, or its diameter if cylindrical, at any horizontal section S,

If W = the superincumbent weight,

T = any thrust inclined at α to the horizon acting at a height y above the section,

b = breadth of section or diameter at S,

V_y = maximum strain at section S,

S = sectional area at S,

then $V_y = \frac{1}{S}\left(\frac{4yT\cos\alpha}{b} \pm W\right)$;

and to avoid tension at the base of the pier where $y=h$, and b becomes b',

$$b' \text{ must be greater than } \frac{4h.T\cos a}{W}.$$

To prevent sliding of any one course upon another, tan β must be less than the coefficient of friction, which for masonry is 0·7.

To prevent crushing, the extreme strain per unit of surface must always at both edges of the pier, and on the line of thrust, be less than the safe crushing resistance of the material, both $v_0 \sec\beta$ and $v_t \sec\beta$ less than R.

For values of R see tables attached to Chapter III. The valuation of the terms throughout a large pier at every variation of section is necessarily lengthy and tedious.

The external conditions under which the sets of terms must be calculated are:—

1st. When both adjacent arches of curved ribs are fully loaded to the extreme that will occur.

2nd. When one adjacent arch or rib is unloaded, and the other is not yet built.

3rd. The same case as the second, but adding temperature stress resulting from a curved rib.

4th. With unbalanced partial moving loads on the two adjacent arches or ribs.

5th. Under the supposition that in extreme flood the water has permeated under the foundation course.

Besides these conditions of ordinary stability the effect of high wind on the pier itself must be anticipated on the whole superstructure, &c., of two adjacent spans above water, and of current below water level. The resultants of each will act on the centre line of the pier; there will then be a total horizontal weather force, acting at some deter-

minable height above the foundation of the pier, which is resisted by a total vertical force, consisting of the weight of the pier and its superstructure.

The direction of the resultant can then be calculated, and the distance from the middle of the pier to its point of application, compared with the whole length of pier, affords evidence of sufficient or insufficient weather stability.

SECTION IV.

PIERS, SUPPORTS, STANCHIONS, &c., SOLID AND BRACED.

Solution Number 1.—*The simple upright shaft or strut of uniform section.*

In a former section a pier treated as a bridge pier of solid material was dealt with under the hypothesis that it resisted both superincumbent weight and thrusts or lateral forces inclined to verticality at any angle; the present case differs from it in being virtually exempt from thrust.

The two effects of superincumbent weight on a simple shaft, or of pressure on a strut applied parallel to its axis of form, are direct compression and bending.

Simple as the problem may appear, it has not yet been rigorously solved in either case, and partly empirical formulæ based upon insufficient experiment constitute at present the sole guide.

The fracture of the shaft is one of the points to be kept in view, but its distortion is equally important in the practical problem, and the two matters become necessarily blended in the consideration of probable failure, as the latter is in some cases a stage preceding fracture, in others the distortion itself constitutes failure.

A shaft may fail without bending under simple compression; secondly, it may give way both under bending and crushing; thirdly, it may yield by bending alone, so as to become practically valueless; fourthly, it may yield under mere transverse stress.

1st. *Direct crushing.*—It is usually believed that a shaft of a length less than 8 diameters (or breadths in square section) fails from crushing alone. Although this limit is generally accepted, there is no reason for any such general limit, as any limit will vary with the nature of material and with the quality of the material of whatever sort it may be. The utility of any limit is doubtful, for a shaft should neither fail from crushing nor from bending.

The ordinary formula of stress and strain under compression is, as before given in the chapter on strains,

$$W=RS, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{Eq. I.}$$

where W is the weight, or longitudinally applied force;
R is the unitary safe resistance to compression of the material,
S is the sectional area compressed,

the same units in weight, pressure, and area, being used throughout the formula.

It is impossible to say to what conditions the applications of this formula is limited.

Rondelet, putting it in the form $W=k.RS$, found that with oaken and pine posts of square section, k varied with the ratio of length to thickness, thus

$\frac{l}{d}$	12	24	36	48	60
k	0·8	0·5	0·33	0·16	0·008

These were probably timbers fixed above and below, as, with

timber-piles entirely embedded in earth to restrain them from bending, the conditions would not have been convenient or precise.

At the other extreme the law of direct crushing fails; with very short lengths, less than two diameters. The limits of its correct application are hence such that in most practical cases the effect of inherent weight of shaft can be neglected. With hollow sections such a law would be less applicable, as failure would be assisted by a tendency to buckle; and, whatever the law of resistance might be in such cases, the resistance would certainly vary with $\frac{t}{d}$, the ratio of effective thickness to total diameter or width.

Hence the conditions under which crushing is applicable in the elementary mode of estimation confine themselves to solid sections, of from about 2 to 8 diameters in length, as a very coarse rule, and up to about 10 diameters in wrought iron, and the less hazardous materials.

2nd. *Partial crushing combined with bending.*—This is the ordinary case for struts, stanchions, and pillars. The empirical formula of Gordon, deduced from those of Hodgkinson and based on the old experiments of Hodgkinson, is used in two forms, and is supposed to hold for all shafts up to 60 diameters in length, and rather beyond.

If W = safe weight or dead load applied at one end,
R = safe resistance to crushing of the material,
S = the sectional area of the shaft or strut,
k = an experimental coefficient.

With fixed ends $$W = \frac{RS}{1 + k.\left(\frac{l}{d}\right)^2}. \quad . \quad . \quad . \quad . \quad \text{Eq. II.}$$

With free ends substitute $4k$ for k; and with one end only

fixed, substitute $1{\cdot}8k$ for k in this formula; the values of k being thus

Material	Timber	Cast iron	Masonry	Wrought iron
k	$\frac{1}{250}$	$\frac{1}{400}$	$\frac{1}{600}$	$\frac{1}{3000}$

With hollow sections d is usually taken as the least width of the rectangle circumscribing the section, and the corresponding radius of gyration (r) of the circumscribing section is employed in the modified formula

$$W = \frac{RS}{1 + k \cdot \dfrac{l^2}{12r^2}}.$$

This modification is usually assigned to Rankine; the values of r for various sections are thus given by him, for use with it, in terms of d.

Section	Value of r
Solid rectangle, least side d,	$\frac{1}{12}d^2$
Thin square cell, least side d,	$\frac{1}{6}d^2$
Thin rectangular cell, sides d and b,	$\frac{1}{12}d^2 \cdot \dfrac{d+3b}{d+b}$
Thin triangular cell, base d,	$\frac{1}{12}d^2$
Solid cylinder, diameter d,	$\frac{1}{16}d^2$
Thin cylindrical cell,	$\frac{1}{8}d^2$
Angle iron of equal ribs, each d,	$\frac{1}{24}d^2$
Angle iron of unequal ribs,	$\frac{1}{12}bd^2(b^2+d^2)$
H iron, breadth of flanges $=b$; " joint area $= B$, web area $= A$	$\frac{1}{12}b^2 \cdot \dfrac{A}{A+B}$
+ cross of equal arms, least dimension $=d$,	$\frac{1}{24}d^2$
Channel iron, area of flanges $=B$, web area $=A$, depth of flange $+\frac{1}{2}$ web-thickness $=h$	$h^2 \cdot \left(\dfrac{B}{12(A+B)} + \dfrac{AB}{4(A+B)}\right)$
Barlow rail, quadrants of radius R,	$\frac{1}{7}R^2$
Pair of Barlow rails, riveted base to base,	$0{\cdot}393R^2$
Circular segment, radius R, length $2R\theta$,	$R^2\left\{\frac{1}{2} + \dfrac{\cos\theta\sin\theta}{2\theta} - \dfrac{\sin^2\theta}{\theta}\right\}$.

In the foregoing methods, both for simple and for partial crushing under longitudinal compression, it is assumed that the resultant pressure acts exactly along the

axis of the shaft, and that the pressure is evenly distributed on the whole section of the shaft.

With uneven distribution the resistance will be reduced, and it further becomes necessary to prevent tension in the section. In this case the least intensity of pressure must not be negative, and the greatest intensity must be less than twice the mean intensity.

Then in the figure, if G be the centre of gravity of the section, C the centre of pressure, at which the mean stress or resultant of the load acts, M the point of application of the greatest intensity of load W, W must be less than $\frac{1}{2}RS$.

Also if GC $=c$, GM $=m$; c must be less than $\frac{I}{mS}$; where $I=$ moment of inertia of the section. The reciprocals of $\frac{I}{mS}$ are given for a few sections in the following table of Rankine.

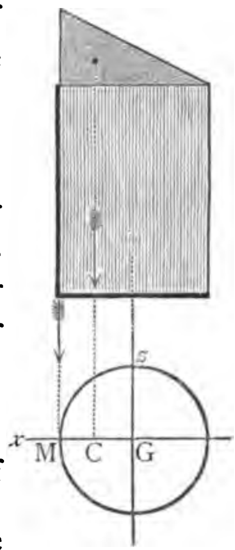


FIGURE I.

Section	Position of neutral axis through G at right angles to C	Value of $\frac{mS}{I}$
Rectangle, sides b and d . .	parallel to b	$\frac{1}{6}d$
Square sided	„ d	$\frac{1}{6}d$
Ellipse, axes b and d . . .	on axis b	$\frac{8}{d}$
Circle, diameter d . . .	on a diameter	$\frac{8}{d}$
Hollow rectangle, b, d, b_1, d_1 .	parallel to b	$6d.\frac{bd-b_1d_1}{bd^3-b_1d_1^3}$
Hollow square, d, d_1 . . .	„ d	$\frac{6d}{d^2+d_1^2}$
Circular ring, d, d_1 . . .	on a diameter	$\frac{8d}{d^2+d_1^2}$

The factor of reduction for unevenly distributed load is applied in the following formula

$$W = \frac{RS}{\left\{1 + k.\frac{l^2}{12r^2}\right\}\left(1 + cm\frac{S}{I}\right)}.$$

A shorter mode of allowing for unequal distribution of load is to use in the foregoing formula the factor

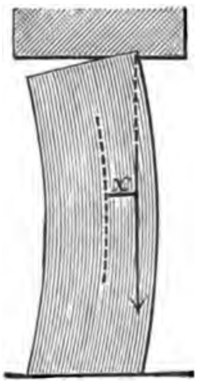


FIGURE 2.

$1 + x.\frac{a}{I}$ instead of $1 + cm\frac{S}{I}$; in which case

a = the extreme axial distance in the section

x = the extreme deviation of the line of mean stress from the axis of figure of the shaft.

Neither of these modes are perfect, and the latter is especially incomplete. The moment of force in an extreme case is not merely a function of x. The leverage of the weight may amount to half the external diameter of the shaft, and assuming the maximum moment to be $\frac{1}{2}Wd$, this should be substituted for W in the general equation for Rupture under compression.

$$Ra\sec\beta + W.\cos\beta\tan\phi = W\sin\beta.$$

The value of β, the angle of rupture, will then correspond to the minimum value of W; but the values of β, ϕ, and R for the material are in this case necessary. Mr. Hodgkinson's values contained in the 11th Report of the British Association are $\beta = 48°$ to $58°$ with various quantities of cast iron; his experiments also showed that when from irregular fixing the deviation of the line of pressure from the axis was $\frac{1}{4}d$, the resistance was lessened by a half.

The whole subject requires fresh investigation, both experimentally and analytically.

3rd. *Distortion under simple bending.*—The flexure of a shaft produced by longitudinal compression cannot be determined in any way, but the flexure that may be sustained by it in a shaft already bent by horizontal pressure can be obtained by solution.

The commoner and usual coarse modes of doing this may be sometimes useful, and are hence given in this section, but the matter of sustained displacement is treated in detail in the paragraphs on Braced Piers, where elastic deformation is more important, from affecting more detail.

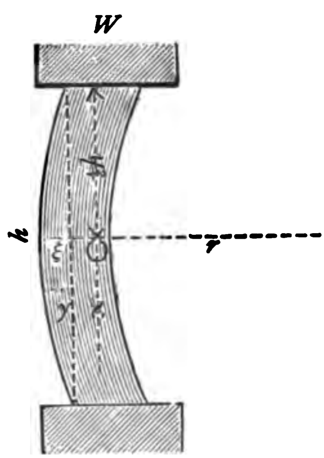


FIGURE 3.

Applying the same method that is used for flexure in girders, but assuming the force sustaining flexure to be W at the top of a shaft, its moment with reference to the point of flexure at the middle of its height h will be $W\xi$, where ξ is the flexure sustained; and this moment will represent the effective force, in whatever direction it may act, whether transversely or otherwise.

The equation of strength will under that hypothesis be $W.\xi = \dfrac{RI}{a}$, where R is the unitary resistance of material,

I is the moment of inertia of the section with reference to neutral axis,

a is the axial distance of its furthest lamina.

But as the flexure is small compared with h, the height, and smaller still compared with r, the radius of curvature, we have $\xi : \frac{1}{2}h :: \frac{1}{2}h : r-\xi$; and $r-\xi=r$;

$\therefore r\xi=\frac{1}{4}h^2$. Also from the consideration of the conditions of

elastic curvature $r=\frac{dE}{R}$, where d is the width of the section, and E is the modulus of elasticity of the material. Combining these conditions, and noticing that $a=\frac{1}{2}d$,

$$W=\frac{RI}{a\xi}=\frac{RI.4r}{ah^2}=\frac{4RI.dE}{aRh^2}=\frac{8EI}{h^2}, \quad . \quad . \quad . \quad . \quad . \quad \text{Eq. III.}$$

a result that may hold within the elastic limits.

The alternative mode of treating sustained flexure in shafts of uniform section consists in putting in the general equation, with deflexions y and abscissæ x,

$Wy=-EI.\delta^2_x y$; solving it and applying coefficients, $y=A\sin\left(x\sqrt{\frac{W}{EI}}\right)+B\left(\cos x\sqrt{\frac{W}{EI}}\right)$; where the first term of the second number does not affect the detail in analysis at the two limits of x; also as the condition that $y=0$, when $x=\frac{1}{2}h$ must hold; we have

$$\cos.\left(\tfrac{1}{2}h\sqrt{\frac{W}{EI}}\right)=0\ ;\ \text{or}\ \tfrac{1}{2}h\sqrt{\frac{W}{EI}}=\tfrac{1}{2}\pi\ ;$$

whence $W=\frac{\pi^2.EI}{h^2}$, or nearly $\frac{10EI}{h^2}$.

This last solution asserts that a far greater weight W would be required to sustain the same flexure as in the former solution: that is to say, if used for the analogous purpose of causation of flexure, this formula asserts less distortion with the same weight; and in practical use errs in weakness. In both methods the inherent weight of the shaft is neglected.

With hollow sections either cellular or radiating, the same method is commonly applied through reduction to a representative hollow rectangle or cylinder, as explained under deflection of girders.

4th. *Yielding under mere transverse stress.*—Taking the

case of a mooring-post, where its own weight is neglected, and the stress of the mooring-chain is applied to a height not exceeding three quarters of the projecting length (l) in the figure. It evidently corresponds to the case of a cantilever set upright, fixed at ground level, and strained transversely by a concentrated load. Hence, if

W be the permissible breaking weight,
R the safe unitary resistance of material to shearing.
S the sectional area strained, which here is the metallic ring,
l the projecting length of post,
d the effective diameter at ground level;

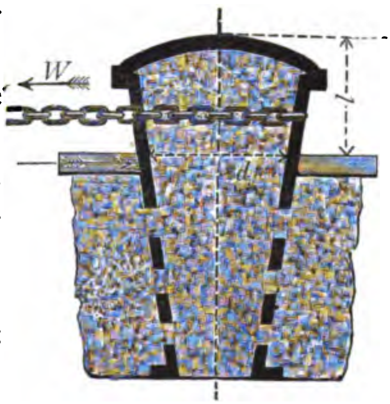


FIGURE 4.—Mooring Post.

where l, d, S, and W and R, are in corresponding units,

$$W.\tfrac{3}{4}l=S.d.R. \quad . \quad . \quad . \quad . \quad \text{Eq. IV.}$$

an equation which establishes the relation and allows the thickness of metal to be determined; or, with a solid post would give the diameter. W is pre-determined by the strength of the mooring-chain used by the ship to be moored, and this is known[1] through Lloyd's regulations; l is generally fixed between 3 and 5 feet, while with a hollow section the total diameter is also nearly arbitrary.

[1] *Lloyd's Regulations for Mooring Chains.*

Ship's Tonnage.	Size of Studded Chains.	Breaking Test (W.)
1 000	$1\frac{3}{4}$ inches	77 tons
1 600	$1\frac{15}{16}$ „	94 „
2 000	$2\frac{1}{16}$ „	107 „
2 500	$2\frac{3}{16}$ „	120 „
3 000	$2\frac{5}{16}$ „	135 „

Last, as to the value of R, when feet are used in all dimensions, and tons as weight-units, the value of R for cast iron is about 400 tons, and for picked good granite R is about 20 tons, for purposes of preliminary calculation.

As very large granite posts would be inconvenient on wharfs, a limit is necessary. The limit about which hollow cast iron becomes necessary is for vessels of 1000 tons.

The concrete filling of the hollow iron post is omitted in the calculation of shearing strain, or resistance.

Solution Number 2.—The braced pier.

Preliminary remarks.—A braced pier may consist of two, four, or more shafts, either vertical or convergent, connected by inclined bracing and horizontal bars; the mode of bracing may be single or double, or inclined braces may be dispensed with; the bars may be either free, fixed, and of compound construction, or may be non-existent; the shafts may be treated as articulated at each tier or perfectly continuous; but the varieties in actual construction are not very great.

The object in employing a braced pier in preference to a cellular or a radiating section is to obtain breadth and lateral stiffness, an advantage in large piers of almost any sort, especially when supporting viaducts or jetties.

The stresses on such piers are vertical loads, horizontal forces, and lateral horizontal stress due to wind-pressure, also the moments of fixture at the top and bottom.

The pier, being elastic, may fail by deformation long before any fracture takes place, hence the possible sorts of distortion have to be considered. Under vertical stress or weight applied at the top as well as inherent weight, the deformation caused is that of subsidence; under lateral or horizontal force necessarily also applied at the top under

usual conditions, the deformation is lateral curvature, with a tendency to overturn. This latter liability has alone received much attention among engineers on account of the facility of determining the resulting strains through simple static resolution ; the complete set of strains can, however, be only determined through the consideration of all the elastic deformations, a lengthy and complicated matter.

As it is evidently faulty to treat an elastic braced construction as rigid, the more correct method will be here adopted ; this will not only determine the strains but also the displacements or deformation at each tier, due to both vertical and horizontal forces of every sort.

Before entering into the general solutions for piers, some preliminary general equations useful in application to them will be deduced from convertible equations with regard to continuous girders.

Preliminary general equations.—A braced pier consisting of two or four shafts connected by bracing may, and generally does, consist in a number of tiers in which the disposition of the parts recurs, each tier being complete in itself. The pier may in the first instance be treated as articulated at each tier ; secondly, the shafts may be treated as continuous ; the analytical transition from one state to

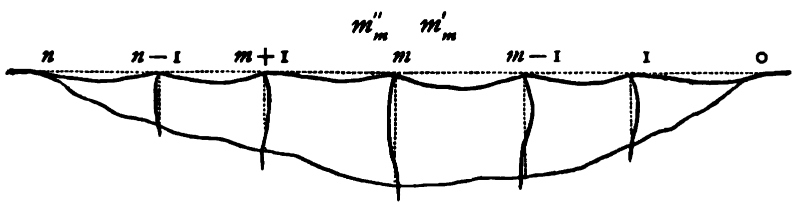


FIGURE 5.

the other may be termed introducing the 'effect of continuity.' As in a horizontal girder the effect of continuity

is reducible, that for a vertical pier may be deduced through it by transformation of the resulting equations.

The following deductions (1) and (2) will therefore *first* apply to a *continuous horizontal girder* supported and fixed on piers and supports, numbered from o on the right to n on the left, m being any intermediate pier (see figure 5).

(1.) Let the girder be *loaded vertically* with a load uniformly distributed per unit of length.

Using the following notation temporarily, let

$M'_m M''_m$ be the bending moments of the girder to right and left of the fixture on the m^{th} pier,

$N_m = M'_m - M''_m$ the moment of reaction of the pier, positive when turning to the right as in figure 6,

l_m and w_m the length of span, and intensity of load on it to the right of the pier m,

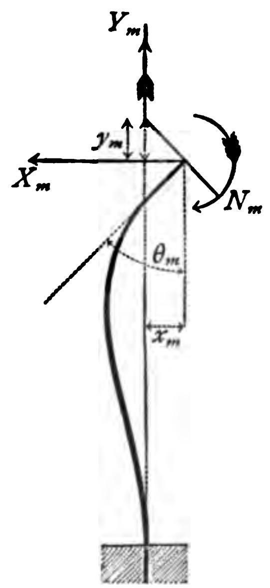


FIGURE 6.

X_m and Y_m the horizontal and vertical reactions of the pier m,

x_m and y_m horizontal and vertical displacements of the top of it,

θ_m angular displacement of top fixed section, positive in the same direction as N_m,

h_m height of the pier m,

S and J the section and moment of inertia of the pier m,

I moment of inertia of the section of the girder,

η and η_1 the moduli of elasticity for the girder and for the pier.

Let the fixture hold at a single point over the axis of the pier; then by known formulæ

$$\left.\begin{aligned}\theta_m &= -\frac{l_m}{6\eta I}\left(M''_{m-1}+2M'_m-\tfrac{1}{4}w_m l_m^2\right)+\frac{y_{m-1}-y_m}{l_m}\,;\\ \theta_m &= +\frac{l_m}{6\eta I}\left(2M''_m+M'_{m+1}-\tfrac{1}{4}w_{m+1}l^2_{m+1}\right)+\frac{y_m-y_{m+1}}{l_{m+1}}\,;\end{aligned}\right\} \quad . \quad (1)$$

Treating the pier as solid and of uniform section, J_m is constant, and as fixed at the base, x_m is in a direction contrary to X_m, in the alignment of the girder,

$$\therefore\ \theta_m=\frac{h_m}{\eta_1 J_m}\left(M'_m-M''_m\right)-\tfrac{1}{2}X_m.h_m\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$X_m=\tfrac{3}{2}.\frac{M'_m-M''_m}{h_m}-\frac{3\eta_1 J_m}{h^3_m}.x_m\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating θ_m and X_m from (1) (2) (3) we have two general equations for continuous girders fixed on elastic piers,

$$\frac{l_m}{6\eta I}\left\{M''_{m-1}+2M'_m-\tfrac{1}{4}w_m l^2_m\right\}+\frac{h_m}{4\eta_1 J}\left(M'_m-M''_m\right)+\frac{3x_m}{2h_m}-\frac{y_{m-1}-y_m}{l_m}=0$$

$$. \quad . \quad . \quad . \quad (4)$$

$$\frac{l_{m+1}}{6\eta I}\left\{2M''_m+M_{m+1}-\tfrac{1}{4}w_{m+1}.l^2_{m+1}\right\}-\frac{h_m}{4\eta_1 J}\left(M'_m-M''_m\right)-\frac{3x_m}{2h_m}+\frac{y_m-y_{m+1}}{l_{m+1}}=0$$

With free supports $M'_m=M''_m=M$; and the sum of these two equations will form a single general equation—a principle that may be reapplied when required.

In (4) the unknowns are M'_m, M''_m, x_m, and y_m. To determine y_m, we have the following relations:

$$y_m=\frac{Y_m h_m}{\eta_1 S_m}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$Y_m=\frac{-M''_{m-1}+M'_m}{l_m}+\frac{M''_m-M'_{m-1}}{l_{m+1}}+\tfrac{1}{2}\left(w_m l_m+w_{m+1}.l_{m+1}\right); \quad (6)$$

To determine x_m; for this x_0, the displacement at the end of the continuous girder at o, must be obtained through the conditions of the girder. Treating it as an elastic solid

acted on by the axial forces X_m and by μY_1, μ being the limiting coefficient of friction suitable, the obtained relations will give X_m, Y_1, and $x_m - x_0$, whence with the aid of (3) and (5) X_m and Y_1 will be eliminated, so that x_m will be obtained in linear functions of M_m, M'_m, M''_m, and x_0.

An alternative approximate solution may be adopted. Neglecting all deformation in the girder, under simple extension $x_m = x_0$ constant throughout. Then Equations (4) will give M' and M'', and (3) will give X_m in linear functions of x_0; x_0 will be obtained from the conditions of horizontal equilibrium of the freely supported girder. For this there are two cases.

First with rigid piers, as of masonry; then $\mu \Sigma Y_1 - \Sigma X_m = 0$; affords a linear equation for obtaining x_0 with the help of (6) eq., provided x_0 exist; this may be discovered by putting $x = 0$ in the above, when μ is known.

Secondly, with braced or flexible piers, then the top section of the pier will adhere to the girder and move with it. The horizontal reaction X_m, see (3), will then be $\frac{3\eta_1 J_1 \cdot x_0}{h_1^3}$; and the condition must exist that x_0 is less than $\frac{\mu Y_1 h_1^3}{3\eta_1 J_1}$; so that the relation from the equilibrium of the girder is

$$\mu \Sigma Y_i + 3\eta_1 x_0 . \Sigma \frac{J_1}{h_1^3} - \Sigma X_m = 0.$$

(2.) Let the girder be acted on by a uniformly distributed *lateral pressure* q per unit of length, as wind pressure acting horizontally, in the direction z.

Also let q_1 be the corresponding uniform lateral pressure on the pier per unit of height. In this case each pier is affected by a concentrated lateral horizontal force at the

top, as well as by q_1, and is subject to flexure and torsion; its reactions destroy continuity in the bending moments of the girder, but the case is generally analogous to the former.

Retaining the symbols η, I, M', M'', h, η_1, J_1, as before, let Y_m and Z_m be the vertical and lateral reactions of the pier,
y_m and z_m the vertical and lateral displacements at the top,
ϕ_m the angular displacement of the top section,
η_2 and J_2 be for the pier the torsive modulus of elasticity and the polar moment of inertia.

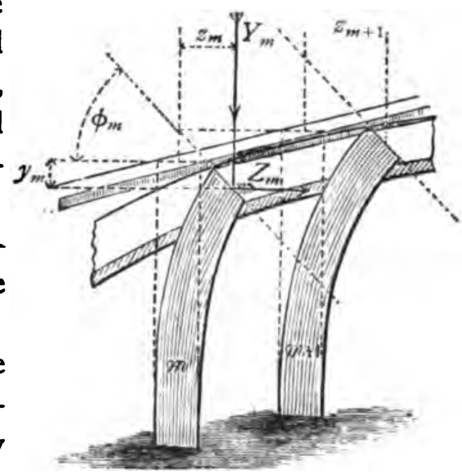


FIGURE 7.—Displacement.

Considering the torsion and bending of the pier affected by Z_m, q_1, and $M'_m - M''_m$, we have at the pier m,

$$\phi_m = \frac{h_m}{\eta_2 J_2}.(M'_m - M''_m); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and $$z_m = \frac{h_m}{\eta_1 J_1}(\tfrac{1}{3} Z_m + \tfrac{1}{8} q_1 h_m); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$Z_m = \frac{-M''_{m-1} + M'_m}{l_m} + \frac{M''_m - M'_{m+1}}{l_{m+1}} + \tfrac{1}{2} q (l_m + l_{m+1}); \quad . \quad . \quad (9)$$

Transforming Eq. (1), putting ϕ for θ, z for x, q for w_m and w_{m+1}; eliminating ϕ_m through (7); the two following equations for the fixed continuous girder result, in which the values of z may be substituted through (8) and (9);

$$\left.\begin{aligned}&\frac{l_m}{6\eta I}\cdot(M''_{m-1}+2M'_m-\tfrac{1}{4}ql_m^2)+\frac{h_m}{\eta_2 J_2}(M'_m-M''_m)-\frac{z_{m-1}-z_m}{l_m}=0\,;\\&\frac{l_{m+1}}{6\eta I}(2M''_m+M'_{m+1}-\tfrac{1}{4}ql^2_{m+1})+\frac{h_m}{\eta_2 J_2}(M'_m-M''_m)+\frac{z_m-z_{m+1}}{l_{m+1}}=0\,;\end{aligned}\right\} \quad (10)$$

When the torsive resistance is neglected, and the girder is freely supported, so that $M'_m=M''_m=M_m$, the above two become one equation:

$$l_m M_{m-1}+2(l_m+l_{m+1})M_m+l_{m+1}\cdot M_{m+1}-6\eta I\left\{\frac{z_{m-1}-z_m}{l_m}-\frac{z_m-z_{m+1}}{l_{m+1}}\right\}=\tfrac{1}{4}q(l^3_m+l^3_{m+1})\,; \quad (11)$$

which, considering (8), contains five successive moments.

When the spans are all equal, a series results in which each equation contains five successive terms in z or Z, which are the sole unknown quantities. In that case (9) and (11) become

$$-M_{m-1}+2M_m-M_{m+1}=Z_m l-ql^2\,;$$

$$M_{m-1}+4M_m+M_{m+1}=\frac{6\eta I}{l^2}(z_{m-1}-2z_m+z_{m+1})+\tfrac{1}{2}ql^2\,;$$

whence $M_m=\frac{1}{6}Z_m l+\frac{\eta I}{l^2}(z_{m-1}-2z_m+z_{m+1})-\frac{1}{12}ql^2\,;$ (12)

Eliminating M by this, one of the two former gives

$${}_{m-1}+4Z_m+Z_{m+1}=\frac{6\eta I}{l^3}(-z_{m-2}+4z_{m-1}+6z_m+4z_{m+1}-z_{m+2})+6ql\,; \quad (13)$$

from this z or Z may be eliminated through (8); and a general equation containing five successive terms in z or Z results.

When h and J remain constant for all the piers, the equation for Z takes the form

$$AZ_{m-2}+(1-4A)Z_{m-1}+(4+6A)Z_m+(1-4A)Z_{m+1}+AZ_{m+2}=6ql;$$

where $A=\frac{2\eta Ih^3}{\eta_1 Jl^3}$; and it is satisfied by $Z=ql$.

General formulæ for continuous piers.

As a pier fixed at its end corresponds to a continuous girder fixed at its end, the effect of continuity in a pier may be deduced from that in a girder, by transformation from horizontality to verticality; hence the foregoing deductions may be transformed.

We will, however, retain y, V, as vertical; x, X, as horizontal; z, Z, as laterally horizontal; in displacements and reactions, however we may transform. The top section of the pier will be numbered 0 corresponding to the right end section of the girder; the bottom section of the pier will be numbered n, and any intermediate one m. The angular displacements θ and ϕ will retain their relative significations in their respective planes of notation after transformation. The moments M in a girder will become H in a pier, the spans l in a girder will become tier-heights h in a pier, and I in a girder becomes J in a pier.

The transformation will apply to series Eq. (11) and (13), which will hold equally true either in the plane xy or in the plane yz, and are applicable when $m=0$, and $m=n$ at the ends, as well as for any intermediate value. Putting them in their complete forms, transformed for piers, Eq. (11) will give $n+1$ equations of the following type. (See Eq. (1) also.)

$$\left.\begin{aligned}
&\theta_0-\frac{x_0-x_1}{h_1}=\frac{1}{6\eta J}(2h_1H_0+h_1H_1-\tfrac{1}{4}qh_1^3);\\
&\frac{x_{m-1}-x_m}{h_m}-\frac{x_m-x_{m+1}}{h_{m+1}}=\frac{1}{6\eta J}\cdot\left\{h_mH_{m-1}+2(h_m+h_{m+1})H_m+h_{m+1}.H_{m+1}-\tfrac{1}{4}q(h^3{}_m+h^3{}_{m+1})\right\};\\
&\frac{x_{n+1}-x_n}{h_n}-\theta_n=\frac{1}{6\eta J}(h_n.H_{n-1}+2h_nH_n-\tfrac{1}{4}qh^3{}_n);
\end{aligned}\right\}\quad\ldots\quad(14)$$

When the tiers are *equal*, a series of the type (13) applicable in $n-1$ equations may be derived from (14); by eliminating H_0 from the two first of the series H_n from the two last, and by substituting for the remaining terms in H their values according to (12), this (12) being applicable to $n-1$ equations, we hence have

$$\left.\begin{aligned}
&7X_1+2X_2=\frac{6\eta J}{h^3}(11x_0-18x_1+9x_2-2x_3-6\theta_0h)+9qh;\\
&X_{m-1}+4X_m+X_{m+1}=\frac{6\eta J}{h^3}(x_{m-2}-4x_{m-1}+6x_m-4x_{m+1}+x_{m+2})+6qh;\\
&2X_{n-2}+7X_{n-1}=\frac{6\eta J}{h^3}(6\theta_nh-2x_{n-3}+9x_{n-2}-18x_{n-1}+11x_n)+9qh;
\end{aligned}\right\}\quad\ldots\quad(15)$$

Also these same equations relating to the pier may be transformed to the plane zy; y remaining vertical; and if horizontal force in either plane be neglected then $q=0$.

These series (14) and (15) express the effect of continuity in the shafts of a pier at the tiers, and will be used in analysis whenever this condition is introduced in a series of nominal tiers.

Further transformation in these will be merely due to any minor change of symbols.

General Solution Number 3.—The Braced Pier of four Converging Shafts.

This type of pier is the most comprehensive one; the solution of this case is due to Allievi, although in some respects it is modified and altered by the author of this book to harmonise with his own solutions. It remains even now inevitably lengthy.

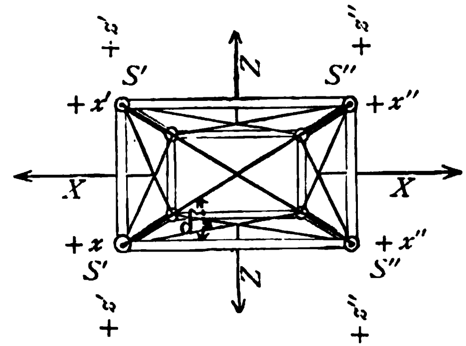


FIGURE 8.—Plan of a convergent tier.

Adopting the following notation,

Let $4W$ and $4Q$ be the vertical and horizontal forces applied at the top.

$4w$ and $4q$, the uniformly distributed vertical and horizontal forces on the shafts.

$2M_m$, the moment of all forces on the part of pier above m with respect to a line e_m.

$2N_m$, the same with respect to a line f_m.

X, Y, Z, the horizontal, vertical, and lateral reactions or total components of strains at any tier.

x, y, z, displacements horizontally, vertically, and laterally, taken positive when measured outwards and downwards.

H', H'', components of bending moment of shafts parallel to xy.

K', K'', components of bending moment of shafts parallel to xz.

V'_m, V''_m, shearing forces immediately above any joints m', m''.

h, the height of any tier of the pier, constant.

b_m, t_m, the breadth and width of the pier at a joint m' m''.

e_m horizontal length between intersections of braces, above m.

f_m horizontal intercepted length at summit of convergence between produced braces of any tier m.

g_m inclined height between b_m and e_m intercepted.

S', S'', sectional areas of shafts on left and right faces.

s', s'', sectional areas of inclined braces on front and rear faces.

c', c'', sectional areas of inclined braces on left and right flanks.

a, sectional areas of horizontal bars on front and rear faces.

a', a'', sectional areas of horizontal bars on left and right flanks.

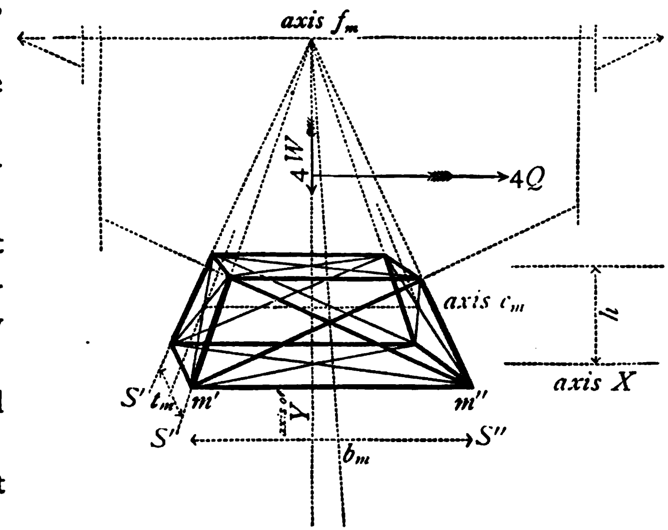


FIGURE 9.—Front view of a convergent tier.

d, sectional areas of internal diagonal braces.

$90^\circ - \alpha$, solid angle formed by a pier face and a pier flank.

β_m, inclination to the face-bissectrix of a face-brace.

β'_m, the projection of β_m on the middle plane.

γ_m, the inclination to the flank-bissectrix of a flank-brace.

δ, the inclination of internal diagonals with face-bar.

ϵ, ϵ_1, the inclinations of a shaft to a face- and to a flank-bissectrix.

ϕ, ϕ_1, the inclinations of a face and of a flank to verticality.

θ the inclinations of a shaft to verticality.

I and r, the moment of inertia and radius of gyration of each shaft with reference to bending in the plane xy.

I_1 and r_1 corresponding quantities with reference to bending in the plane xz.

J and r_2 analogous quantities for the whole pier with reference to the plane xy.

η and η_1 moduli of elasticity of shafts and braces.

R, R_1, R_2, &c. resistances or strains on sections; these as well as the moduli accompany their own sections.

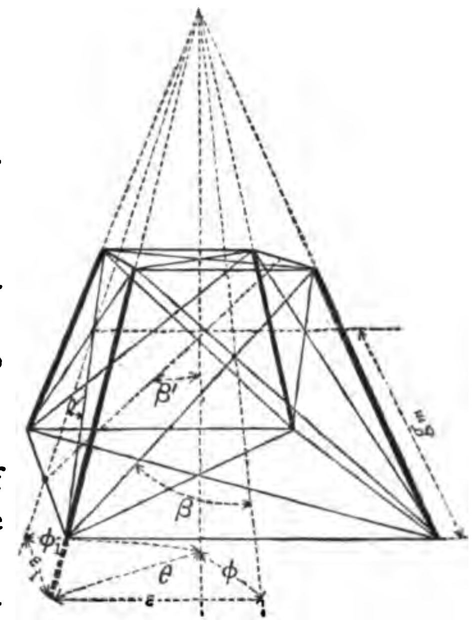


FIGURE 10.—The same view.

First, the strains on the various members may be represented in terms of the displacements effected by the stresses, at any tier $m' \, m''$, thus

$$\left.\begin{array}{l}
RS'_m = \left\{(x'_{m-1} - x'_m)\sin\epsilon + (y'_{m-1} - y'_m)\cos\epsilon + (z'_{m-1} - z'_m)\sin\epsilon_1\right\}\dfrac{\eta_1 S_m}{h_m}\cos\theta\,; \\[2ex]
Rs'_m = \left\{(z''_{m-1} - z'_m)\cos\beta_m . \sin\phi + (y''_{m-1} - y'_m)\cos\beta_m\cos\phi - (x''_{m-1} + x'_m).\sin\beta_m\right\}\dfrac{\eta s_m}{h_m}.\cos\beta_m.\cos\phi\,; \\[2ex]
Rc'_m = \left\{(x'_{m-1} - x'_m)\cos\gamma_m\sin\phi_1 + (y'_{m-1} - y'_m)\cos\gamma\cos\phi_1 - (z'_{m-1} + z'_m)\sin\gamma_m\right\}.\dfrac{\eta c_m}{h}.\cos\gamma_m.\cos\phi_1. \\[2ex]
RS''_m,\ Rs''_m, \text{ and } Rc''_m \text{ will be expressed in exactly corresponding terms and signs excepting} \\
\qquad\qquad\qquad \text{that each displacement will be conversely dashed.} \\[2ex]
Ra_m = (x'_m + x''_m)\dfrac{\eta a_m}{b_m}\,; \quad Ra'_m = 2z'_m.\dfrac{\eta a'_m}{t_m}\,; \quad Ra''_m = 2z''_m\dfrac{\eta a''_m}{t_m}\,; \\[2ex]
Rd_m = \left\{(x'_m + x''_m)\cos\delta + (z'_m + z''_m)\sin\delta\right\}\dfrac{\eta d_m}{b_m}.\cos\delta\,; \\[2ex]
\qquad = \left\{(x'_m + x''_m)\cos\delta + (z'_m + z''_m)\sin\delta\right\}\dfrac{\eta d_m}{t_m}.\sin\delta\,;
\end{array}\right\}\text{(I.)}$$

The relations of the angular quantities may be thus expressed for future use :

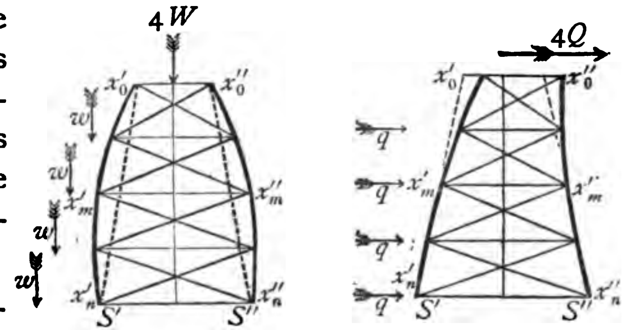
$$\left.\begin{array}{ll}
\cos\theta=\cos\epsilon\cos\phi=\cos\epsilon_1\cos\phi_1; & \\
\sin\epsilon=\sin\phi_1\cos\epsilon_1; & \sin\epsilon_1=\sin\phi.\cos\epsilon; \\
\sin(\beta_m+\epsilon)=\frac{b_m}{h_m}.\cos\beta_m.\cos\theta; & \sin(\gamma_m+\epsilon_1)=\frac{t_m}{h_m}.\cos\gamma_m.\cos\theta; \\
\sin(\beta_m-\epsilon)=\frac{b_{m-1}}{h_m}.\cos\beta_m.\cos\theta; & \sin(\gamma_m-\epsilon_1)=\frac{t_{m-1}}{h_m}.\cos\gamma_m.\cos\theta; \\
\tan\beta_m \quad =\frac{b_m+b_{m-1}}{2h_m}.\cos\phi; & \tan\gamma_m \quad =\frac{t_m+t_{m-1}}{2h_m}.\cos\phi_1; \\
\tan\epsilon \quad =\frac{b_m-b_{m-1}}{2h_m}.\cos\phi; & \tan\epsilon_1 \quad =\frac{t_m-t_{m-1}}{2h_m}.\cos\phi_1; \\
\tan\phi \quad =\frac{t_m-t_{m-1}}{2h_m}; & \tan\phi_1 \quad =\frac{b_m-b_{m-1}}{2h_m}; \\
\tan\beta_m.\tan\epsilon=\tan\gamma_m.\tan\epsilon_1; & \cos a \quad =\frac{\cos\phi}{\cos\epsilon_1}=\frac{\cos\phi_1}{\cos\epsilon};
\end{array}\right\} \quad \cdots\cdots \quad \text{(II.)}$$

The strains and the displacements may be now subdivided and grouped in two sets, due respectively to the influence of vertical and of horizontal force.

Under vertical force, the displacements will be $\frac{1}{2}(x'+x'')$, &c., or half sums, and the strains will be equal compressions in homologous inclined braces, and equal tensions in internal diagonals and external bars.

Under horizontal force the displacements will be $\frac{1}{2}(x''-x')$, &c., or half differences, and the

strains equal but contrary in each homologous pair of members on a face at right angles to the plane of action, while there will be no strain on the bars and internal diagonals. The half sums and half differences which apply in all strains, displacements, bending moments, and components will now be symbolised *in one term only*, with the simple subscript w or q; and each set will require a separate solution.



Subsidence. Overturning.

FIGURES 11 and 12.—Deformation on a single face.

Before proceeding, we will add some compound terms to the notation for future brevity, all of which are to apply to the same tier m.

$$\left.\begin{aligned}
&A_1=\frac{\eta s.\cos^2\beta.\cos^2\phi}{\eta S\cos^2\theta+\eta s\cos^2\beta.\cos^2\phi+\eta c\cos^2\gamma\cos^2\phi_1}; \quad A_3=\frac{\eta c\cos^2\gamma\cos^2\phi_1}{\eta S\cos^2\theta+\eta c\cos^2\gamma\cos^2\phi_1};\\
&A_2=\frac{\eta c.\cos^2\gamma.\cos^2\phi_1}{\eta S\cos^2\theta+\eta s\cos^2\beta.\cos^2\phi+\eta c\cos^2\gamma\cos^2\phi_1}; \quad A_4=\frac{1}{\eta S\cos^2\theta+\eta c\cos^2\gamma\cos^2\phi_1};\\
&B_1=A_1.\eta S\cos^2\theta;\quad B_2=A_2.\eta S\cos^2\theta;\quad B_3=A_3.\eta S\cos^2\theta;\quad B_4=A_1.\eta c\cos^2\gamma.\cos^2\phi_1;\\
&C_1=\eta a.\left(\frac{h}{b}\right)^2;\qquad C_2=\eta a'\left(\frac{h}{t}\right)^2;\qquad C_3=\eta d.\left(\frac{h}{b}\right)^2\cos^2\delta=\eta d\left(\frac{h}{t}\right)^2\sin^2\delta;\\
&D=\frac{h_m}{h_{m+1}}=\frac{b_m}{b_{m+1}}=\frac{t_m}{t_{m+1}};
\end{aligned}\right\} \quad . \quad . \quad \text{(III.)}$$

Strains and displacements due to vertical force.—Equations of equilibrium. Treating the part of the pier above a pair of joints $m'\,m''$, we have by equilibrium vertically

$$(RS_m+RS''_m)\cos\theta+(Rs'_m+Rs''_m)\cos\beta_m\cos\phi_1+(Rc'_m+Rc''_m).\cos\gamma_m\cos\phi_1=2\,W+2\Sigma_1^{m-1}.(w_1+w_m)=2\,W_m.$$

Applying this to a series of values of m, from 1 to n, and making use of I., II., and III., we have generally an equation in which the subscript w should be applied throughout. (This subscript will be neglected safely if we bear in mind that this section is devoted to results of vertical force, under *single* terms.)

$$y_{m-1}-y_m=\frac{W_m h_m}{\text{divisor of } A_1}+\left\{\frac{b_m}{h_m}.A_1-\tan\phi_1\right\}x_{m-1}+\left\{\frac{b_{m-1}}{h_m}.A_1+\tan\phi_1\right\}x_m+\left\{\frac{t_m}{h_m}.A_2-\tan\phi\right\}z_{m-1}+\left\{\frac{t_{m-1}}{h_m}.A_2+\tan\phi\right\}.z_m\,; \quad \text{(IV.)}$$

Projecting the strains on the braces that meet in m' on the normal to the plane of the flank $S'\,S'$; also those on the braces meeting in m'' on the normal to the plane of the flank $S''\,S''$, we have

$$\left.\begin{aligned} X'_m&=\{Rs'_m.\sin\overline{\beta_m-\epsilon}+Rs''_{m+1}\sin\overline{\beta_{m+1}+\epsilon}\}\cos\alpha-\{Ra_m+Rd_m\cos\delta\}.\cos\phi_1\,;\\ X''_m&=\{Rs''_m.\sin\overline{\beta_m-\epsilon}+Rs'_{m+1}.\sin\overline{\beta_{m+1}+\epsilon}\}\cos\alpha-\{Ra_m+Rd_m.\cos\delta\}.\cos\phi_1\,;\end{aligned}\right\}\quad .\;.\;.\;.\;\text{(V.)}$$

Projecting also the same strains on the normal to the plane of the face $S'\,S''$, we have

$$\left.\begin{aligned} Z'_m&=\{Rc'_m\sin\overline{\gamma_m-\epsilon_1}+Rc'_{m+1}.\sin\overline{\gamma_{m+1}+\epsilon_1}\}\cos\alpha-\{Ra'_m+Rd_m\sin\delta\}.\cos\phi\,;\\ Z''_m&=\{Rc''_m.\sin\overline{\gamma-\epsilon_1}+Rc''_{m+1}.\sin\overline{\gamma_{m+1}+\epsilon_1}\}\cos\alpha-\{Ra''_m+Rd_m\sin\delta\}\cos\phi\,;\end{aligned}\right\}\quad .\;.\;.\;.\;.\;\text{(VI.)}$$

Adding the two equations (V.), also the two of (VI.), after transforming by (I.), the differences $y_{m-1}-y_m$ and y_m-y_{m+1} may be eliminated through (IV.). The two resulting equations divided respectively by $\cos\phi_1$ and $\cos\phi$, and simplified through II. and III. will become

$$\begin{aligned}\frac{X_{mw}}{\cos\phi_1} &= \frac{b_{m-1}}{h_m}.(A_1)_m W_m + \frac{b_{m+1}}{h_{m+1}}.(A_1)_{m+1} W_{m+1} - \frac{b_{m-1}.b_m}{h^3_m}.(B_1+B_4)_m x_{m-1w} - \left\{\frac{b^2_{m-1}}{h^3_m}(B_1+B_4)_m + \frac{2b^2_m}{h^3_m}(C_1+C_3)_m\right. \\ &\left. + \frac{b^2_{m+1}}{h^3_{m+1}}.(B_1+B_4)_{m+1}\right\} x_{mw} - \frac{b_m.b_{m+1}}{h^3_{m+1}}(B_1+B_4)_{m+1} x_{m+1w} + \frac{b_{m-1}.t_m}{h^3_m}(B_4)_m.z_{m-1w} + \left\{\frac{b_{m-1}.t_{m-1}}{h^3_m}.(B_4)_m - \frac{2b_m t_m}{h^3_m}.(C_3)_m\right. \\ &\left. + \frac{b_{m+1}.t_{m+1}}{h^3_{m+1}}(B_4)_{m+1}\right\} z_{mw} + \frac{b_{m+1}.t_m}{h^3_{m+1}}(B_4)_{m+1} z_{m+1w} \quad \ldots\ldots \quad \text{(VII.)}\end{aligned}$$

and

$$\begin{aligned}\frac{Z_{mw}}{\cos\phi} &= \frac{t_{m-1}}{h_m}.(A_2)_m W_m + \frac{t_{m+1}}{h_{m+1}}(A_2)_{m+1}.W_{m+1} - \frac{t_{m-1}.t_m}{h^3_m}.(B_2+B_4)_m z_{m-1w} - \left\{\frac{t^2_{m-1}}{h^3_m}.(B_2+B_4)_m + \frac{2t^2_m}{h^3_m}(C_2+C_3)_m\right. \\ &\left. + \frac{t^2_{m+1}}{h^3_{m+1}}(B_2+B_4)_{m+1}\right\} z_{mw} - \frac{t_m.t_{m+1}}{h^3_{m+1}}.(B_2+B_4)_{m+1}.z_{m+1w} + \frac{t_{m-1}.t_m}{h^3_m}.(B_4)_m x_{m-1w} + \left\{\frac{t_{m-1}.b_{m-1}}{h^3_m}.(B_4)_m - \frac{2t_m.b_m}{h^3_m}.(C_3)_m\right. \\ &\left. + \frac{t_{m+1}.b_{m+1}}{h^3_{m+1}}\right\} x_{mw} + \frac{b_m t_{m+1}}{h^3_{m+1}}.(B_4)_{m+1}.x_{m+1w}\,; \quad \ldots\ldots \quad \text{(VIII.)}\end{aligned}$$

the fundamental relations.

1. With articulated tiers. In this case the conditions of equilibrium at the joints m' m'', in directions perpendicular to the planes of the faces of the pier, give

$$X'_m=\tfrac{1}{2}(w_m+w_{m+1}).\sin\phi_1+q_m\cos\phi_1;\quad\text{and}\quad Z'_m=\tfrac{1}{2}(w_m+w_{m+1}).\sin\phi;$$
$$X''_m=\tfrac{1}{2}(w_m+w_{m+1})\sin\phi_1-q_m\cos\phi_1;\quad\text{and}\quad Z''_m=\tfrac{1}{2}(w_m+w_{m+1})\sin\phi.$$

Adding the two first and the two second, dividing by $\cos\phi_1$ and $\cos\phi$ respectively, the results are

$$\frac{X_{mw}}{\cos\phi_1}=\tfrac{1}{2}(w_m+w_{m+1}).\tan\phi_1;\quad\text{and}\quad\frac{Z_{mw}}{\cos\phi}=\tfrac{1}{2}(w_m+w_{m+1})\tan\phi.$$

Also (VII.) and (VIII.) afford two series each of $n-1$ equations applied from when $m=1$ up to $m=n-1$, of the general form

$$(\kappa_1)_m x_{m-1w}+(\kappa_2)_m x_{mw}+(\kappa_3)_m x_{m+1w}-(\sigma_1)_m z_{m-1w}-(\sigma_2)_m z_{mw}-(\sigma_3)_m z_{m+1w}=\frac{b_{m-1}}{h_m}.(A_1)_m W_m+\frac{b_{m+1}}{h_{m+1}}(A_1)_{m+1}.W_{m+1}$$
$$-\tfrac{1}{2}(w_m+w_{m+1}).\tan\phi_1;\quad\ldots\ldots\quad\text{(IX.)}$$

$$(\chi_1)_m z_{m-1w}+(\chi_2)_m z_{mw}+(\chi_3)_m z_{m+1w}-(\zeta_1)_m x_{m-1w}-(\zeta_2)_m x_{mw}-(\zeta_3)_m x_{m+1w}=\frac{t_{m-1}}{h_m}(A_2)_m.W_m+\frac{t_{m+1}}{h_m}.(A_2)_{m+1}.W_{m+1}$$
$$-\tfrac{1}{2}(w_m+w_{m+1})\tan\phi;\quad\ldots\ldots\quad\text{(X.)}$$

at the ends $x_{ow}=x_{nw}=z_{ow}=z_{nw}=0$.

The above coefficients may be reduced if required at full length. But in the special case when in every tier the sections of the braces are constant, and β and γ are constant, we have

then A_1, A_2, B_1, B_2, B_4. C_1, C_2, C_3, D, constant in every tier; so that the subscripts for tiers in IX. and X. may be suppressed, and IX. and X. become with similar tiers simply

$$D(B_1+B_4)b_m x_{m-1w}+\{(D+D^2)(B_1+B_4)+2C_1+2C_3\}b_m x_{mw}+D^2(B_1+B_4)b_m x_{m+1w}-DB_4 t_m z_{m-1w}$$
$$-\{(D+D^2).B_4-2C_3\}t_m.z_{mw}-D^2B_4 t_m z_{m+1w}=h_m^2\{A_1(DW_m+W_{m-1})-\tfrac{1}{4}(1-D).(w_m+w_{m+1})\}; \quad (\text{IX.}a)$$
$$D(B_2+B_4)t_m z_{m-1w}+\{(D+D^2)(B_2+B_4)+2C_2+2C_3\}t_m z_{mw}+D^2(B_2+B_4)t_m z_{m+1w}-DB_4 b_m x_{m-1w}$$
$$-\{(D+D^2).B_4-2C_3\}b_m x_{mw}-D^2B_4 b_m x_{m+1w}=h_m^2\{A_2(DW_m+W_{m+1})-\tfrac{1}{4}(1-D)(w_m+w_{m+1})\}; \quad (\text{X.}a)$$

2. With continuous shafts fixed at the ends.

When the tiers are all of equal height, and I and I_1 for the shafts are constant, as well as their sectional areas, the mode of obtaining x_w and z_w is analogous to that of the preceding Preliminary Equations, of which equation (15) can be applied here. The bending moments of the shafts must be projected on planes passing through them normal to the faces of the pier.

In transforming (Prel. Eq. 15) h will become $h \sec \theta$; $x_n=\theta_n=0$. Also when dealing with the flanks $S'S$ and $S''S''$; $q=w \sin \phi_1$; $I=Sr^2$; with the shaft S'; $X_m=X'_m$, and $x_m=-x'_m \cos\phi_1$; and with the shaft S''; $X_m=X''_m$, and $x_m=-x''_m \cos \phi_1$.

When dealing with the faces $S'S''$; $q=w.\sin\phi$; $x_0=\theta_0=0$; $I=I_1=Sr_1^2$; with the shaft S'; $X_m=Z'_m$; $x_m=-z'_m \cos\phi$; and with the shaft S''; $X_m=Z''_m$, and $x_m=-z'' \cos\phi$.

Adding together the two series of equations in each pair thus obtained, and noticing that $\frac{4wh}{\cos\theta}=4w_0$ the constant weight of a tier, we shall have two series of equations resulting, which, divided by $\cos\phi_1$ and $\cos\phi$ respectively, will take the form,

$$\frac{X_{m-1w}}{\cos\phi_1}+\frac{4X_{mw}}{\cos\phi_1}+\frac{X_{m+1w}}{\cos\phi_1}=\frac{6\eta S\cos^3\theta . r^2}{h^3}.(x_{m-2w}-4x_{m-1w}+6x_{mw}-4x_{m+1w}+x_{m+2w})+6w_0\tan\phi_1\,; \qquad \text{(XI.)}$$

$$\frac{Z_{m-1w}}{\cos\phi}+\frac{4Z_{mw}}{\cos\phi}+\frac{Z_{m+1w}}{\cos\phi}=\frac{6\eta S\cos^3\theta . r_1^2}{h^3}.(z_{m-2w}-4z_{m-1w}+6z_{mw}-4z_{m+1w}-z_{m+2w})+6w_0\tan\phi\,; \quad . \qquad \text{(XII.)}$$

Eliminating the terms in the first members of these through VII. and VIII., in which suppressing the subscript of h we may put $W_m=W+w_0(m-\frac{1}{2})$; two series will result giving the displacements x_w and z_w, in the following form with some new coefficients,

$$E_1x_{m-2w}+E_2x_{m-1w}+E_3x_{mw}+E_4x_{m-1w}+E_5x_{m-2w}-F_1z_{m-2w}-F_2z_{m-1w}-F_3z_{mw}-F_4z_{m-1w}-F_5z_{m-2w}$$
$$=(G_1W+G_2w_0)h^2\,; \quad . \quad . \quad . \quad . \qquad \text{(XIII.)}$$

$$E'_1z_{m-2w}+E'_2z_{m-1w}+E'_3z_{mw}+E'_4z_{m-1w}+E'_5z_{m-2w}-F'_1x_{m-2w}-F'_2x_{m-1w}-F'_3x_{mw}-F'_4x_{m-1w}-F'_5x_{m-2w}$$
$$=(G'_1W+G'_2w_0)h^2\,; \quad . \quad . \quad . \quad . \qquad \text{(XIV.)}$$

The values of these coefficients, all which apply to the tier m, as regards vertical force, are as follow,

$$\left.\begin{array}{l}
(E_1)_m=b_{m-2}.b_{m-1}(B_1+B_4)_{m-1}+6\eta S\cos^3\theta . r^2\,;\\
(E_2)_m=b^2_{m-2}(B_1+B_4)_{m-1}+b_m(4b_{m-1}+b_m)(B_1+B_4)_m+2b^2_{m-1}(C_1+C_3)_{m-1}-24\eta S\cos^3\theta . r^2\,;\\
(E_3)_m=b_{m-1}(4b_{m-1}+b_m)(B_1+B_4)_m+b_{m+1}(b_m+4b_{m+1})(B_1+B_4)_{m+1}+8b^2_m(C_1+C_3)_m+36\eta S\cos^3\theta . r^2\,;\\
(E_4)_m=b_m(b_m+4b_{m+1})(B_1+B_4)_{m+1}+b^2_{m+2}.(B_1+B_4)_{m+2}+2b^2_{m+1}(C_1+C_3)_{m+1}-24\eta S\cos^3\theta . r^2\,;\\
(E_5)_m=b_{m+1}.b_{m+2}.(B_1+B_4)_{m+2}+6\eta S\cos^3\theta . r^2\,;\\
\text{the dashed coefficients } E' \text{ are correspondingly the same, after putting } B_2 \text{ for } B_1 \text{ and } C_2 \text{ for } C_1.
\end{array}\right\}$$

$$\left.\begin{aligned}
&(F_1)_m=t_{m-1}.b_{m-2}.(B_4)_{m-1}\,;\\
&(F_2)_m=t_{m-2}.b_{m-2}.(B_4)_{m-1}+t_m(4b_{m-1}+b_m)(B_4)_m-2t_{m-1}.b_{m-1}.(C_3)_{m-1}\,;\\
&(F_3)_m=t_{m-1}(4b_{m-1}+b_m)(B_4)_m+t_{m+1}(b_m+4b_{m+1})(B_4)_{m+1}-8t_mb_m(C_3)_m\,;\\
&(F_4)_m=t_m.(b_m+4b_{m+1})(B_4)_{m+1}+t_{m+2}.b_{m+2}.(B_4)_{m+2}-2t_{m+1}.b_{m+1}.(C_3)_{m+1}\,;\\
&(F_5)_m=t_{m+1}b_{m+2}.(B_4)_{m+2}\,;\\
&\text{the dashed coefficients } F' \text{ are correspondingly the same, putting everywhere } t \text{ for } b, \text{ and } b \text{ for } t.\\
&(G_1)_m=b_{m-2}.(A_1)_{m-1}+(4b_{m-1}+b_m)(A_1)_m+(b_m+4b_{m+1})(A_1)_{m+1}+b_{m+2}.(A_1)_{m+2}\,;\\
&(G_2)_m=(m-\tfrac{3}{2})b_{m-2}.(A_1)_{m-1}+(m-\tfrac{1}{2})(4b_{m-1}+b_m).(A_1)_m+(m+\tfrac{3}{2})b_{m+2}(A_1)_{m+2}\\
&\qquad+(m+\tfrac{1}{2})(b_m+4b_{m+1})(A_1)_{m+1}-12h\tan\phi_1\,;
\end{aligned}\right\}\text{(XV.)}$$

the dashed coefficients G' are the same, putting everywhere t for b, and A_2 for A_1;

The half sums H_{mw} and K_{mw} can be obtained from the transformation of Preliminary Eq. (14) by projecting the bending moments of the shafts on planes passing through them normal to the faces of the pier. The change of symbols will be the same as that for XI. and XII. Adding together each pair of relations thus obtained, and multiplying the two results respectively by $\cos\epsilon_1$ and $\cos\epsilon$, also making use of II., we have

$$\left.\begin{aligned}
&H_{mw}.\cos\epsilon_1=\tfrac{1}{8}h.X_{mw}\sec\phi_1-\eta S\cos^3\theta.\frac{r^2}{h^2}\left(x_{m-1w}-2x_{mw}+x_{m+1w}\right);\\
&K_{mw}.\cos\epsilon=\tfrac{1}{8}h.Z_{mw}.\sec\phi-\eta S\cos^3\theta.\frac{r_1^2}{h^2}\left(z_{m-1w}-2z_{mw}+z_{m+1w}\right);
\end{aligned}\right\}\quad\text{(XVI.)}$$

At the top sections,

$$\left.\begin{aligned} H_{ow}.\cos\epsilon_1 &= -\tfrac{1}{12}h.X_{1w}\sec\phi_1 - \eta S\cos^3\theta.\frac{r^2}{h^2}\left(5x_{1w}-x_{2w}\right); \\ K_{ow}.\cos\epsilon &= -\tfrac{1}{12}hZ_{1w}.\sec\phi - \eta S\cos^3\theta.\frac{r_1^2}{h^2}\left(5z_{1w}-z_{2w}\right); \end{aligned}\right\}$$

and analogously at the bottom sections also fixed H_{nw} and K_{nw} will follow this last form.

This rigorous solution holds so far with tiers of equal height, but the expressions for displacement, either in IX and X or in XIII and XIV are impracticable in application.

Approximation.—With similar tiers an approximation will hold rigorously with respect to W, but merely approximately with regard to w the inherent weight, which is different in successive tiers. This will enable us to arrive at the displacements and strains, as here following.

For purposes of approximation, putting

$$\left.\begin{aligned} &U_1=(D+D^2)(B_1+B_4)+C_1+C_3; \qquad U_2=(D+D^2)(B_2+B_4)+C_2+C_3; \\ &U_3=C_3-(D+D^2)(B_4); \qquad T_1=\frac{A_1.U_2-A_2U_3}{U_1U_2-U_3^2}; \text{ and } T_2=\frac{A_2U_1-A_1U_3}{U_1U_2-U_3^2}; \end{aligned}\right\} \quad . \; . \; . \; . \; \text{(XVII.)}$$

$$\text{then } x_{mw}=T_1.\frac{h^2_m}{b_m}.\tfrac{1}{2}(1+D).W; \text{ and } z_{mw}=T_2.\frac{h^2_m}{t_m}.\tfrac{1}{2}(1+D)W; \; . \; . \; . \; . \; . \; . \; . \; . \; . \; \text{(XVIII.)}$$

With the aid of this, and with Equations I., II., III., IV., the strains due to vertical force, constant in every tier, will have the following values,

$$\left.\begin{array}{ll} RS_w=\{1-T_1C_1-T_2C_2-2C_3(T_1+T_2)\}W\sec\theta; & Rs_w=\{T_1C_1+C_3(T_1+T_2)\}W.\sec\beta\sec\phi; \\ Rc_w=\{T_2C_2+C_3(T_1+T_2)\}W\sec\gamma.\sec\phi_1; & Ra_w=\ T_1C_1.2W.\tan\beta\sec\phi; \\ Ra'_w=\ T_2C_2.2W.\tan\gamma.\sec\phi_1; & Rd_w\begin{cases}=C_3(T_1+T_2).2W.\tan\beta.\sec\delta\sec\phi; \\ =C_3(T_1+T_2).2W.\tan\gamma\operatorname{cosec}\delta\sec\phi_1;\end{cases} \end{array}\right\}\text{(XIX.)}$$

Strains, &c., due to horizontal force, consisting of half differences of strains and displacements before mentioned, treated as single terms with the subscript q, being due to $4Q$, $4q$, $2M$, $2N$.

Noticing now the symbols mentioned before, which will now come into use, and their values,

$$e_m=\frac{2b_m.b_{m-1}}{b_m+b_{m-1}};\qquad f_m=\frac{2b_m.b_{m-1}}{b_m-b_{m-1}};\qquad g_m=\frac{b_m\cos(\beta'_m-\phi_1)}{2\sin\beta'_m\cos\epsilon_1};\text{ also that }\tan\beta'_m=\tan\beta_m.\sec\phi.$$

Equations of Equilibrium. Treating a top portion of pier, down to the joints $m'\ m''$ of any tier, its moments of rotation, first with respect to the axis e_m and secondly with respect to the axis f_m, will be thus,

$$\{(RS''_m-RS'_m)\cos\epsilon_1+(Rc''_m-Rc'_m)\cos\gamma_m\}\tfrac{1}{2}e_m.\cos\phi_1=M_m+(H'_m-H''_m)\cos\epsilon_1+(V'_m-V''_m)g_m.\cos\epsilon_1;$$
$$(Rs''_m-Rs'_m).\tfrac{1}{2}f_m.\cos\beta_m.\cos\phi=N_m+(H'_m-H''_m)\cos\epsilon_1-(V'_m-V''_m).\tfrac{1}{2}b_m.\operatorname{cosec}\phi_1.$$

But as $V'_m=(-H'_{m-1}+H'_m)\frac{1}{h^m}.\cos\theta$; $V''_m=(-H''_{m-1}+H_m)\frac{1}{h^m}.\cos\theta$; these two equations may after eliminating g_m, and general reduction with the aid of II. &c., take the forms

$$RS_{mq}.\cos\theta+Rc_{mq}\cos\gamma_m\cos\phi_1=\frac{L_m}{e_m}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \text{(XX.)}$$

$$Rs_{mq}.\cos\beta_m\cos\phi=\frac{L'_m}{f_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XXI.)}$$

where L and L' have the following values,

$$\left.\begin{array}{l} \dfrac{L_m}{e_m}=\dfrac{M_m}{e_m}-\left\{1+\dfrac{b_{m-1}}{h_m}.\sin\phi_1\cos\phi_1\right\}.\dfrac{H_{m-1q}.\cos\epsilon_1}{b_{m-1}}-\left\{1-\dfrac{b_m}{h_m}.\sin\phi_1\cos\phi_1\right\}.\dfrac{H_{mq}.\cos\epsilon_1}{b_m}; \\ \dfrac{L'_m}{f_m}=\dfrac{N_m}{f_m}+\dfrac{H_{m-1q}.\cos\epsilon_1}{b_{m-1}}-\dfrac{H_{mq}.\cos\epsilon_1}{b_m}; \end{array}\right\} \quad . \quad . \quad \text{(XXII.)}$$

Transforming XX and XXI through I., two series result, applicable to tiers from $m=1$ to $m=n$, of the general form,

$$y_{m-1q}-y_{mq}=\frac{h_m.L_m}{(\eta S_m\cos^3\theta+\eta c_m\cos^3\gamma\cos^3\varphi_1).e_m}+\left(\frac{t_m(A_3)_m}{h_m}-\tan\phi\right)z_{m-1q}+\left(\frac{t_{m-1}.(A_3)_m}{h_m}+\tan\phi\right)z_{mq}-(x_{m-1q}-x_{mq})\tan\phi_1; \quad \text{(XXIII.)}$$

$$x_{m-1q}-x_q=\left\{\frac{h_m.L'_m}{(\eta s_m\cos^3\beta\cos^3\varphi)f_m}+y_{m-1q}+y_{mq}+(z_{m-1q}+z_{mq}).\tan\phi\right\}.\cot\gamma_m.\cos\phi; \quad . \quad . \quad . \quad . \quad \text{(XXIV.)}$$

Also subtracting the two equations (VI) after transformation through (I.), and from the relation obtained eliminating through XXIII the terms in y, the result after reduction and dividing by $\cos\phi$ will be

$$\frac{Z_{mq}}{\cos\phi}=\frac{t_{m-1}(A_3)_m.L_m}{h_m e_m}+\frac{t_{m+1}(A_3)_{m+1}.L_{m+1}}{h_{m+1}.e_{m+1}}-\frac{t_{m-1}.t_m.(B_3)_m}{h^3_m}.z_{m-1q}-\left\{\frac{t^2_{m-1}(B_3)_m}{h^3_m}+\frac{2t^2_m(C_2)_m}{h^3_m}+\frac{t^2_{m+1}(B_3)_{m+1}}{h^3_{m+1}}\right\}z_m$$
$$-\frac{t_m.t_{m+1}}{h^3_{m+1}}.(B_3)_{m+1}.z_{m+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XXV.)}$$

the fundamental formulæ being XX, XXI, XXIII, XXIV, XXV.

1. With articulated tiers. In this case

$L_m = M_m$; $L'_m = N_m$; $Z_{mq} = 0$;

and XXV. affords the series for determining z_q in the general form,

$$(\kappa_1)_m z_{m-1q} + (\kappa_2)_m z_{mq} + (\kappa_3)_m z_{m+1q} = \frac{t_{m-1}.(A_3)_m M_m}{h_m.e_m} + \frac{t_{m+1}.(A_3)_{m+1}.M_{m+1}}{h_{m+1}.e_{m+1}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(XXVI.)}$$

which applies to tension on the faces $S'S'$ and $S''S''$.

When z, &c., are known, y, &c., can be obtained through series XXIII and XXIV, by eliminating $x_{m-1} - x_m$ from the corresponding pairs, and the result will take the general form,

$$\frac{y_{m-1q}}{v_{m-}} - \frac{y_{mq}}{b_m} = \frac{h_m.M_m}{(\eta S\cos^3\theta + \eta c\cos^3\gamma\cos^3\phi_1)e^2_m} - \frac{h_m.N_m}{(\eta s\cos^3\beta\cos^3\varphi)f^2_m} + \frac{t_{m-1}.t_m.(A_3)_m}{e_m h_m}\left\{\frac{z_{m-1q}}{t_{m-1}} + \frac{z_{mq}}{t_m}\right\} - \tan\phi.\left\{\frac{z_{m-1q}}{b_{m-1}} - \frac{z_{mq}}{b_m}\right\}.$$

Observing that $y_{nq} = 0$, the terms in y can be obtained from this by successive summation. Also the terms in x_q can be similarly obtained through XXIV. At length XX and XXI give

$$RS_{mq}.\cos\epsilon_1 + Rc_{mq}\cos\gamma_m = \frac{M_m}{e_m}.\sec\phi_1; \text{ and } \quad Rs_{mq}\cos\beta_m = \frac{N_m}{f_m}.\sec\phi \quad . \; . \; . \; . \; . \; . \; . \quad \text{(XXVII.)}$$

Approximation.—If the tiers are similar, and the sections of the four corresponding members in each tier are constant; an approximate mode may be adopted, so as to divide the effect of $\frac{M_m}{e_m}$ between the two members. Under these conditions XXVI becomes

$$D.B_3 z_{m-1q} + \{(D + D^2).B_3 + 2C_2\} z_{mq} + D^2.B_3 z_{m+1q} = \frac{A_3 h_m^2}{t_m b_m}.\tfrac{1}{2}(1 + D).(M_m + M_{m+1}).$$

Also by neglecting q_m, M_m becomes $M+Q(e_m-b_0)\cot\phi_1$; and $b_0=D_m.b_m$; $e_m=\dfrac{2D.b_m}{1+D}$.

Putting $O=\dfrac{A_8}{(D+D^2).B_8+C_2}$; then XXVI is satisfied by the following value,

$$z_{mq}=O.\frac{h^2{}_m}{t_m b_m}.\tfrac{1}{4}(1+D).\{M_m+M_{m+1}\} \quad \ldots \ldots \quad \text{(XXVIII.)}$$

Also from (I.) we obtain by reduction,

$$RS_{mq}=\left\{(x_{m-1q}-x_{mq})\tan\phi_1+y_{m-1q}-y_{mq}+(z_{m-1q}-z_{mq})\tan\phi\right\}\frac{\eta S\cos^2\theta}{h_m}. \quad \ldots \ldots \quad \text{(XXIX.)}$$

In this the terms $(x_{m-1q}-x_{mq})\tan\phi_1+y_{m-1q}-y_{mq}$ may be replaced by means of XXIII. after modifying the terms according to the hypothesis of similar tiers. In the resulting expression for z_q, the unknowns can be eliminated by XXIX, so that after reduction

$$\left.\begin{array}{ll}
 & RS_{mq}=(1-OC_2)\dfrac{1}{e_m}.M_m\sec\theta; \\
\text{from XXVII.} & Rc_{mq}=OC_2.\dfrac{1}{e_m}.M_m\sec\gamma.\sec\phi_1; \\
\text{from I. and XXIX.} & Rd'_{mq}=OC_2.\dfrac{1}{b_m}.\left(M_m+M_{m+1}\right)\tan\gamma.\sec\phi_1; \\
\text{also} & Rs_{mq}=\dfrac{1}{f_m}.N_m\tan\beta\sec\phi; \quad \text{and } Ra_{mq}=Rd_{mq}=0
\end{array}\right\} \quad . \quad \text{(XXX.)}$$

These approximate results may be utilised in the more rigorous solution by partially replacing the subscripts of the coefficients, and applying them to the case when the tiers are nearly similar.

2. With continuous shafts fixed at both ends. In this case it is apparently impossible to obtain any direct explicit solution holding with regard to horizontal forces and moments, even under the assumptions of similar tiers, and of equality of section in corresponding braces; but an implicit equation may be obtained, which will be practically useful with numerical quantities.

Before entering into the insoluble general equations applicable, we will briefly give an approximate mode of treatment.

Approximation.—Adopting a coefficient, P, deduced from the consideration of $\frac{I_m}{J_m}$;

$$P_m = \frac{e^2_m . \cos^2 \phi_1}{e^2_m . \cos^2 \phi_1 + 4[1 - O_m(C_2)_m . r^2_m]},$$

where r_m is the radius of gyration of the shafts in the tier m; this coefficient is simply applied to the whole of the results of XXX., with the subscripts for the tier m reapplied in all the coefficients, thus

$$RS_{mq} = P_m \{1 - O_m(C_2)_m\} \frac{1}{e_m} . M_m \sec \theta;$$

so also with Rc, Ra', and Rs; Ra and Rd remaining 0; when, as before, $2M_m$ and $2N_m$ are the moments of force acting down to the tier m, with regard to the axes e_m and f_m respectively parallel to the plane of action.

General Treatment.—Reverting now to the general treatment which yields equations that are insoluble, but still useful as regards the effects of fixture, as well as in other respects when

numerical quantities are applied. The determination of H_q and z_q proceeds by applying the preliminary equations of continuity (14 and 15) to the deflexion of the shafts projected on planes passing through them normal to the pier-faces and flanks.

(1.) *Normal to $S'S'$ and $S''S''$ the flanks,*
in the transformation, h becomes $h_m \sec\theta$; $J=Sr^2$; q becomes $w.\sin\phi_1$; $x_n=\theta_n=0$;

at the shaft S', we have $x_m=y'_m\sin\phi_1-x'_m\cos\phi_1$; and $M_m=H'_m$;
at the shaft S'', we have $x_m=y''_m\sin\epsilon_{1m}-x''_m\cos\epsilon_{1m}$; and $M_m=H''_m$;

and by subtracting the two series, dividing by $\cos\epsilon$, and reducing to *single* terms according to position, a series results in the general form

$$\left\{\frac{x_{m-1q}-x_{mq}}{h_m}-\frac{x_{mq}-x_{m+1q}}{h_{m+1}}\right\}.\cos\phi_1-\left\{\frac{y_{m-1q}-y_{mq}}{h_m}-\frac{y_{mq}-y_{m+1q}}{h_{m+1}}\right\}\sin\phi_1=\frac{\cos\theta}{6\eta S\cos^3\theta.r^2}\times$$
$$\left\{h_m.H_{m-1q}+2h_m H_{mq}+2h_{m-1}.H_{mq}+h_{m+1q}.H_{m+1q}\right\}; \quad \text{(XXXI.)}$$

By eliminating the terms in y through XXIII., this becomes

$$\frac{x_{m-1q}-x_{mq}}{h_m}-\frac{x_{mq}-x_{m+1q}}{h_{m+1}}=\frac{\cos\phi_1\cos\theta}{6\eta S\cos^3\theta.r^2}\left\{h_m H_{m-1q}+2H_{mq}(h_m+h_{m-1})+h_{m+1q}H_{m+1q}\right\}+\sin\phi_1\cos\phi_1\left[(A_4)_m.\frac{L_m}{e_m}\right.$$
$$-(A_4)_{m+1}.\frac{L_{m+1}}{e_{m+1}}-\frac{1}{h_m}\left(\frac{t_m(A_3)_m}{h_m}-\tan\phi\right)z_{m-1q}+\left\{\frac{t_{m-1}.(A_3)_m}{h^2_m}+\frac{\tan\phi}{h_m}-\frac{t_{m+1}(A_3)_{m+1}}{h^2_{m+1}}+\frac{\tan\phi}{h_{m+1}}\right\}.z_{mq}$$
$$\left.-\left\{\frac{t_m(A_3)_{m+1}}{h^2_{m+1}}+\frac{\tan\phi}{h_{m+1}}\right\}z_{m+1q}\right]; \quad \ldots\ldots \quad \text{(XXXI.}a\text{)}$$

Now we may transform XXIV by multiplying each of its equations by $\frac{\tan\beta_m}{\cos\phi_1}$, and by then subtracting each equation from the preceding, noticing that

$\frac{\tan\beta_m}{\cos\phi_1}=\frac{b_m}{h_m}-\tan\phi_1=\frac{b_{m-1}}{h_m}+\tan\phi_1$, the resulting series will take the form

$$\left\{\frac{x_{m-1q}-x_{mq}}{h_m}-\frac{x_{mq}-x_{m+1q}}{h_{m+1}}\right\}b_m=\frac{h_m L'_m}{(\eta s\cos^3\beta.\cos^3\phi)_m f_m}-\frac{h_{m+1}.L'_{m+1}}{(\eta s\cos^3\beta\cos^3\phi.f)_{m+1}}+(x_{m-1q}-x_{m+1q})\tan\phi_1+y_{m-1q}-y_{m+1q}$$
$$+(z_{m-1q}-z_{m+1q}).\tan\phi\,; \quad \ldots \quad \text{(XXIV.}a\text{)}$$

From this equation the terms in x_q in the first member may be eliminated through (XXXI.a); and the terms in y_q and x_q in the second member may be eliminated through the equation resulting from adding together the m^{th} and $m+1^{th}$ of series (XXIII.); hence after reduction it will take the following complicated form, involving a set of new coefficients.

$$(E'_1)_m.h_m.\frac{H_{m-1q}}{b_{m-1}}+\left\{(E'_2)_m h_m+(E''_2)_m.h_{m+1}\right\}\frac{H_{mq}}{b_m}+(E''_1)_m.h_{m+1}.\frac{H_{m+1q}}{b_{m+1}}=h_m\eta S\cos^3\theta\left\{(A_4)_m.\frac{M_m}{e_m}.\left(1-\frac{b_m}{h_m}\sin\phi_1\cos\phi_1\right)\right.$$
$$\left.+\frac{N_m}{f_m(\eta s\cos^3\beta\cos^3\phi)_m}\right\}+h_{m+1}.\eta S\cos^3\theta\left\{(A_4)_{m+1}.\frac{M_{m+1}}{e_{m+1}}\left(1+\frac{b_m}{h_{m+1}}\sin\phi_1\cos\phi_1\right)+\frac{N_{m+1}}{f_{m+1}(\eta s\cos^3\beta\cos^3\phi)_{m+1}}\right\}$$
$$+\eta S\cos^3\theta\left\{\frac{(F'_1)_m}{h_m}.z_{m-1q}+\left(\frac{(F'_2)_m}{h_m}+\frac{(F''_2)_m}{h_{m+1}}\right).z_{mq}+\frac{(F''_1)_m}{h_{m+1}}.z_{m+1q}\right\}; \quad \ldots \quad \text{(XXXII.)}$$

These coefficients are with regard to horizontal force

$$\left.\begin{aligned}
(E'_1)_m &= \frac{b_{m-1}.b_m}{6r^2}.\cos^2\phi_1 - \left(\frac{\eta S\cos^3\theta}{\eta s\cos^3\beta.\cos^3\phi}\right)_m + \eta S\cos^3\theta(A_4)_m.\left\{1-\frac{b_m}{h_m}.\sin\phi_1.\cos\phi_1\right\} \\
&\quad .\left(1+\frac{b_{m-1}}{h_m}.\sin\phi_1\cos\phi_1\right); \\
(E''_1)_m &= \frac{b_m.b_{m+1}}{6r^2} - \text{\&c. same terms with subscript } m+1 \text{ for } m; \\
(E'_2)_m &= \frac{b^2_m}{3r^2}.\cos^2\phi_1 + \left(\frac{\eta S\cos^3\theta}{\eta s\cos^3\beta\cos^3\phi}\right)_m + \eta S\cos^3\theta.(A_4)_m.\left\{1-\frac{b_m}{h_m}.\sin\phi_1\cos\phi_1\right\}^2; \\
(E''_2)_m &= \frac{b^2_m}{3r^2}.\cos^2\phi_1) + \text{\&c. same terms with subscript } m+1 \text{ for } m; \\
(F'_1)_m &= t_m.(A_3)_m.\left\{1-\frac{b_m}{h_m}.\sin\phi_1\cos\phi_1\right\} + b_m.\frac{\sin\epsilon\sin\epsilon_1}{\cos^2\epsilon_1}; \\
(F''_1)_m &= t_m.(A_3)_{m+1}.\left\{1+\frac{b_m}{h_{m+1}}\sin\phi_1\cos\phi_1\right\} + b_m.\frac{\sin\epsilon.\sin\epsilon_1}{\cos^2\epsilon_1}; \\
(F'_2)_m &= t_{m-1}.(A_3)_m.\left\{1-\frac{b_m}{h_m}.\sin\phi_1\cos\phi_1\right\} - b_m\frac{\sin\epsilon.\sin\epsilon_1}{\cos^2\epsilon_1}; \\
(F''_2)_m &= t_{m+1}(A_3)_{m+1}\left\{1+\frac{b_m}{h_{m+1}}.\sin\phi_1\cos\phi_1\right\} - b_m\frac{\sin\epsilon.\sin\epsilon_1}{\cos^2\epsilon_1};
\end{aligned}\right\} \quad . \quad \text{(XXXIII.)}$$

With similar tiers and equal sections of corresponding braces, and rejecting a term in q, the Equation XXXII becomes,

$$D(E'_1)_m.\frac{H_{m-1q}}{b_{m-1}} + \left\{D(E'_2)_m + (E''_2)_m\right\}.\frac{H_{mq}}{b_m} + (E''_1)_m.\frac{H_{m\ 1q}}{b_{m+1}} = \frac{\eta S\cos^3\theta(A_4)_m}{e_{m+1}}.\left\{M_m.\left(1-\frac{b_m}{h_m}.\sin\phi_1\cos\phi_1\right)\right.$$
$$\left.+M_{m+1}\left(1+\frac{b_m}{h_m}.\sin\phi_1\cos\phi_1\right)\right\} + \frac{\eta S\cos^3\theta}{h_m.h_{m+1}}\left\{(F'_1)_m\, z_{m-1q} + [(F'_2)_m + D(F''_2)_m]z_{mq} + D(F''_1)_m\, z_{m-1q}\right\}; \text{(XXXII.}a)$$

which cannot be explicitly expressed, but is useful when numerical quantities are employed.

(2) *Projection normal to the face $S'S''$.*—In this, the Preliminary Equation 15 is used, and applied to XXV. after transformation, and subtracting the two series thus obtained, we have

$$\{Z_{m-1q}+4Z_{mq}+Z_{m+1q}\}\sec\epsilon_1=\frac{6\eta S\cos^3\theta.r_1^2}{h^3}(z_{m-2q}-4z_{m-1q}+6z_{mq}-4z_{m+1q}+z_{m+2q}) \quad . \quad . \quad . \quad \text{(XXXIV.)}$$

which, like Prel. 15 involves the hypothesis of equal tiers.

Also Equations (XXV.) and (XXXIV.) give a series for H_q and z_q of the general form

$$(G_1)_m z_{m-2q}-(G_2)_m z_{m-1q}+(G_3)_m z_{mq}-(G_4)_m z_{m-1q}+(G_5)_m z_{m-2q}=h^2\Big[t_{m-2}.(A_3)_{m-1}.\frac{L_{m-1}}{e_{m-1}}+(4t_{m-1}+t_m)(A_3)_m.\frac{L_m}{e_m}$$
$$+t_{m+2}.(A_3)_{m+2}.\frac{L_{m+2}}{e_{m+2}}+(t_m+4t_{m+1}).(A_3)_{m+1}.\frac{L_{m+1}}{e_{m+1}}\Big]; \quad . \quad . \quad . \quad \text{(XXXV.)}$$

where the coefficients G used with horizontal forces are thus,

$$\left.\begin{aligned}
(G_1)_m&=t_{m-2}.t_{m-1}.(B_3)_{m-1}+6\eta S\cos^3\theta.r_1^2;\\
(G_2)_m&=t^2_{m-2}.(B_3)_{m-1}+t_m(4t_{m-1}+t_m)(B_3)_{m-1}+2t^2_{m-1}.r_{1m-1}-24\eta S\cos^3\theta.r_1^2;\\
(G_3)_m&=t_{m-1}(4t_{m-1}+t_m).(B_3)_m+t_{m+1}(t_m+4t_{m+1})(B_3)_{m+1}+8(t_2r_1)_m+36\eta S\cos^3\theta.r_1^2;\\
(G_4)_m&=t_m(t_m+4t_{m+1}).(B_3)_{m+1}+t^2_{m+2}.(B_3)_{m+2}+2(t^2_{m+1}.r_1)_{m+1}-24\eta S\cos^3\theta r_1^2;\\
(G_5)_m&=t_{m+1}.t_{m+2}.(B_3)_{m+2}+6\eta S\cos^3\theta.r_1^2;
\end{aligned}\right\} \quad . \quad . \quad \text{(XXXVI.)}$$

It appears that (XXXV.) will not yield explicit values of H_{mq}. Suppressing q_m, and putting approximately $L_m=P_m.M_m=P_m\{M+2Qh(m-\frac{1}{2})\}$; the second member of (XXXV.) will take the form

$$\left.\begin{aligned}
&(u_1)_m.Mh^2+(u_2)_m.2Qh^3, \text{ where}\\
&(u_1)_m=t_{m-2}.(P.A_3)_{m-1}+\{4t_{m-1}+t_m\}.(PA_3)_m+\{t_m+4t_{m+1}\}.(PA_3)_{m+1}+t_{m+2}.(PA_3)_{m+2};\\
&(u_2)_m=(m-\tfrac{3}{2}).t_{m-2}.(A_3)_{m-1}+(m-\tfrac{1}{2})\{4t_{m-1}+t_m\}.(A_3)_m+(m+\tfrac{3}{2})(t.A_3)_{m+2}\\
&\qquad+(m+\tfrac{1}{2})\{t_m+4t_{m+1}\}(A_3)_{m+1}-12h\tan\epsilon.
\end{aligned}\right\} \quad . \quad . \quad . \quad \text{(XXXV.}a\text{)}$$

Equations (XXXII.a) and (XXXV.a) are of some use when numerical quantities are employed: but (with abstract quantities) the strains due to horizontal forces can only be directly obtained through the approximate mode before mentioned on page 314.

Strains due to combined forces of all sorts.—These are as follow under the limitations before mentioned in the approximation under the two separate sets of forces.

$$RS'_m \text{ or } RS''_m = \{1-(T_1C_1)_m-(T_2C_2)_m-2(C_3)_m[T_1+T_2]_m\}.W_m\sec\theta \mp \{1-(OC_2)_m\}\frac{1}{e_m}.(PM)_m.\sec\theta;$$

$$Rs'_m \text{ or } Rs''_m = \{(T_1C_1)_m+(C_3)_m.[T_1+T_2]_m\}\, W_m\sec\beta_m.\sec\phi \mp \frac{1}{f_m}(PN)_m\sec\beta_m.\sec\phi_m;$$

$$Rc'_m \text{ or } Rc''_m = \{(T_2C_2)_m+(C_3)_m(T_1+T_2)_m\}\, W_m\sec\gamma_m\sec\phi_1 \mp (OC_2)_m.\frac{1}{e_m}.(PM)_m\sec\gamma_m.\sec\phi_1;$$

$$Ra_m = (T_1C_1)_m\{W_m+W_{m+1}\}\tan\beta_m\sec\phi;$$

$$Ra'_m \text{ or } Ra''_m = (T_2C_2)_m\{W_m+W_{m+1}\}\tan\gamma_m\sec\phi_1 \mp (OC_2)_m\frac{1}{b_m}\{W_m+W_{m+1}\}.\tan\gamma_m.\sec\phi_1;$$

$$Rd_m = D_m\{T_1+T_2\}_m.\{W_m+W_{m+1}\}\tan\beta_m.\sec\delta.\sec\phi; = D_m\{T_1+T_2\}_m.\{W_m+W_{m+1}\}.\tan\gamma_m\operatorname{cosec}\delta\sec\phi_1;$$

Special case.—When the convergent pier is square in plan, the formulæ are slightly simplified, from putting $t=b$; $\gamma=\beta$; $c=s$; $a=a'=a''$; $\phi=\phi_1$; $\epsilon=\epsilon_1$; $A_1=A_2$; $B_1=B_2$; $C_1=C_2$; $\delta=45°$; $x_{mw}=z_{mw}=T_1\frac{h^2_m}{b_m}.W_m.\frac{1}{2}(1+D)$; and $T_1=T_2$; this will reduce the terms in the coefficients.

In reducing these, care must be taken not to confound the coefficients E, F, G used with vertical forces with the separate set E, F, G, used with horizontal force, and to distinguish between the two cases of equal tiers and similar tiers in the mixed solution.

General Solution Number 4.—*The braced pier of four vertical shafts.*

When the plan of the pier is rectangular and the shafts are vertical, the bracing remaining similar, this is evidently

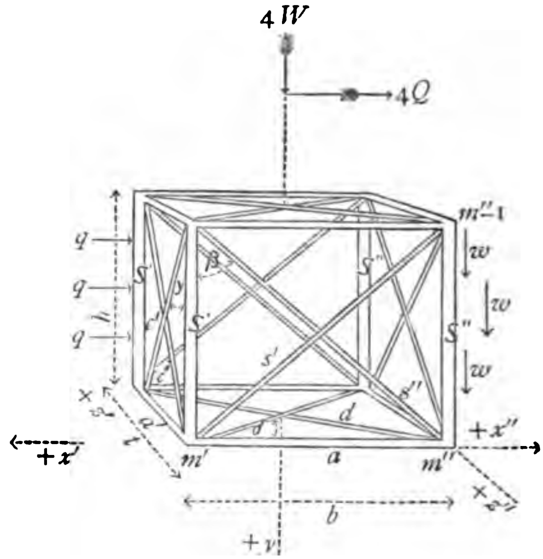


FIGURE 13.—Tier above m' m' of vertical rectangular pier.

a special case of the rectangular pier with four converging shafts, where $\alpha=\theta=0°$; $\epsilon=\epsilon_1=0°$; and $\phi=\phi_1=0$; the cosines and the secants of these angles, each $=1$; also $\beta'=\beta$; $e_m=b$; $f_m=\infty$; $g_m=\frac{1}{2}h$; b and t are constant; $\frac{h}{b}=\operatorname{cotg}\beta$; $\frac{h}{t}=\operatorname{cotg}\gamma$; $D=1$.

We may therefore adopt the whole of the results of General Solution No. 3, after reduction on account of these conditions. Not only so, but the approximate results of Solution No. 3 will then hold more closely with vertical tiers, as the limits there introduced assumed a partial verticality in each separate tier.

The simplified series of coefficients will now be,

$$A_1=\frac{\eta s\cos^3\beta}{\eta S+\eta s\cos^3\beta+\eta c\cos^3\gamma}; \qquad B_1=A_1.\eta S;$$

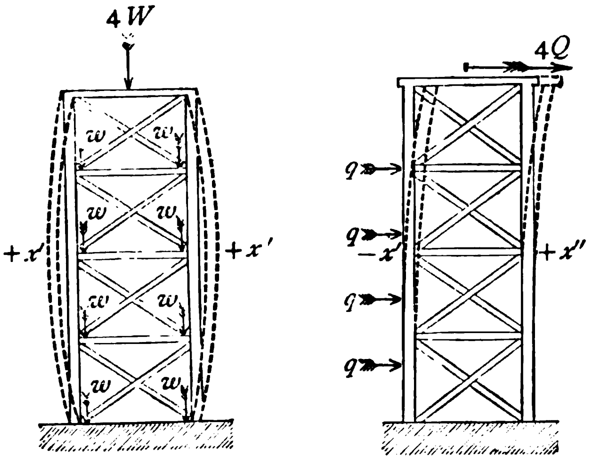
$$A_2=\frac{\eta c\cos^3\gamma}{\eta S+\eta s\cos^3\beta+\eta c\cos^3\gamma}; \qquad B_2=A_2.\eta S;$$

$$A_3=\frac{\eta c\cos^3\gamma}{\eta S+\eta c\cos^3\gamma}; \qquad B_3=A_3.\eta S;$$

$$A_4=\frac{1}{\eta S+\eta c\cos^3\gamma}; \qquad B_4=A_1.\eta c.\cos^3\gamma;$$

$$C_1=\eta a\left(\frac{h}{b}\right)^3=\eta a\operatorname{cotg}^3\beta; \qquad C_3=\eta d\cot^3\beta\cos^3\delta;$$

$$C_2=\eta a'\left(\frac{h}{t}\right)^3=\eta c\operatorname{cotg}^3\gamma; \qquad D=1.$$



Displacement
under vertical stress. under horizontal stress.

FIGURES 14 AND 15.

Under vertical stress.—For the equation of vertical displacements,

$$y_{m-1w}-y_{mw}=\frac{A_1W_mh}{\eta s\cos^3\beta}+\frac{A_1.b}{h}\left\{x_{m-1w}+x_{mw}+\left\{\frac{A_2.t}{h}\left\{z_{m-1w}+z_{mw}\right\}\right\}\right\}; \tag{IV.}$$

the fundamental relations, or values of X_{mw} and Z_{mw} are

$$X_{mw}=\frac{A_1.b}{h}(W_m+W_{m+1})-\frac{b^2}{h^3}\Big\{(B_1+B_4)x_{m-1w}+2(B_1+B_4+C_1+C_3)x_{mw}$$

$$+(B_1+B_4)x_{m+1w}\Big\}+\frac{bt}{h^3}\Big\{B_4 z_{m-1w}+2(B_4-C_3)z_{mw}+B_4 z_{m+1w}\Big\};\text{(VII.)}$$

$$Z_{mw}=\frac{A_2.t}{h}(W_m+W_{m+1})-\frac{t^2}{h}\Big\{(B_2+B_4)z_{m-1w}+2(B_2+B_4+C_2+C_3)z_{mw}$$

$$+(B_2+B_4)z_{m+1w}\Big\}+\frac{bt}{h^3}\Big\{B_4 x_{m-1w}+2(B_4-C_3)x_{mw}+B_4 x_{m+1w}\Big\};$$

(VIII.)

treating the pier as articulated at each tier; $X_{mw}=0$, and $Z_{mw}=0$; and the values of series of x_w and z_w are thus

$$b(B_1+B_4)x_{m-1w}+2b(B_1+B_4+C_1+C_3)x_{mw}+b(B_1+B_4)x_{m+1w}-tB_4'z_{m-1w}$$
$$-2t(B_4-C_3)z_{mw}-t.B_4 z_{m+1w}=A_1h^2(W_m+W_{m+1})\,; \quad . \quad . \quad . \quad . \quad . \quad \text{(IX.)}$$
$$t(B_2+B_4)z_{m-1w}+2t(B_2+B_4+C_2+C_3)z_{mw}+t(B_2+B_4)z_{m+1w}-bB_4x_{m-1w}$$
$$-2b(B_4-C_3)x_{mw}-bB_4x_{m+1w}=A_2h^2(W_m+W_{m+1})\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(X.)}$$

treating the shafts as continuous and fixed at the ends, the values of the series of x_w and z_w are

$$E_1x_{m-2w}+E_2x_{m-1w}+E_3x_{mw}+E_2x_{m+1w}+E_1x_{m+2w}-F_1z_{m-2w}-F_2z_{m-1w}$$
$$-F_3z_{mw}-F_2z_{m+1w}-F_1z_{m+2w}=12A_1h^2b(W+mwh); \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.)}$$
$$E'_1z_{m-2w}+E'_2z_{m-1w}+E'_3z_{mw}+E'_2z_{m+1w}+E'_1z_{m+2w}-F'_1x_{m-2w}-F'_2x$$
$$-F_3'x_{mw}-F_2'x_{m+1w}-F_1'x_{m+2w}=12A_2h^2t(W+mwh)\,; \quad . \quad . \quad . \quad . \quad \text{(XIV.)}$$

where $E_1=(B_1+B_4)b^2+6\eta Sr_1^2$; $\qquad F_1=btB_4$;

$E_2=\{6(B_1+B_4)+2(C_1+C_3)\}b^2-24\eta Sr_1^2$; $\quad F_2=bt(6B_4-2C_3)$;

$E_3=\{10(B_1+B_4)+8(C_1+C_3)\}b^2+36\eta Sr_1^2$; $F_3=bt(10B_4-8C_3)$;

and the values of the dashed coefficients E' and F' correspond, putting B_2 for B_1, C_2 for C_1, r_2 for r_1, t for b. Also

$$H_{mw}=\tfrac{1}{6}hX_{mw}-\eta S\frac{r_1^2}{h^2}(x_{m-1w}-2x_{mw}+x_{m+1w})\,;$$

$$K_{mw}=\tfrac{1}{6}hZ_{mw}-\eta S\frac{r_2^2}{h^2}(z_{m-1w}-2z_{mw}+z_{m+1w})\,; \quad . \quad . \quad . \quad \text{(XVI.)}$$

In the explicit approximation

$$U_1=2(B_1+B_4)+(C_1+C_3);\ U_2=2(B_2+B_4)+C_2+C_3;\ U_3=C_3-2B_4;$$

$$T_1=\frac{A_1U_1-A_2U_3}{U_1U_2-U_3^2};\ T_2=\frac{A_2U_1-A_1U_3}{U_1U_2-U_3^2};$$

the corresponding values of x_{mw} and z_{mw} at all tiers except the two extremities will be

$$x_{mw}=T_1.\frac{h^2}{b}.\tfrac{1}{2}(W_m+W_{m+1});\ z_{mw}=T_2.\frac{h^2}{t}.\tfrac{1}{2}(W_m+W_{m+1});\quad \text{(XVIII.)}$$

and the strains due to vertical stress are

$$\begin{aligned}
RS_{mw}&=\{1-T_1C_1-T_2C_2-2C_3(T_1-T_2)\}W_m;\\
Rs_{mw}&=\{T_1C_1+C_3(T_1+T_2)\}W_m\sec\beta;\\
Rc_{mw}&=\{T_2C_2+C_3(T_1+T_2)\}W_m\sec\gamma;\\
Ra_{mw}&=T_1C_1(W_m+W_{m+1}).\tan\beta;\\
Rd'_{mw}&=T_2C_2(W_m+W_{m+1}).\tan\gamma;\\
Rd_{mw}&=C_3(T_1+T_2)(W_m+W_{m+1}).\tan\beta\sec\delta;\\
&=C_3(T_1+T_2)(W_m+W_{m+1}).\tan\gamma\operatorname{cosec}\delta;\quad .\quad \text{(XIX.)}
\end{aligned}$$

Under horizontal stress.—The forces are

$$V'_m=\frac{-H'_{m-1}+H'_m}{h}+\tfrac{1}{2}qh;\ V''_m=\frac{-H''_{m-1}+H''_m}{h}-\tfrac{1}{2}qh;$$

$$\begin{aligned}
L_m&=M_m-H_{m-1q}-H_{mq};\qquad L'_m=Q_mh+H_{m-1q}-H_{mq};\\
M_m&=M+Qh(m-\tfrac{1}{2})+qh^2(m^2-m+\tfrac{1}{2});\\
Q_mh&=Q+qh(m-\tfrac{1}{2});
\end{aligned}$$

and the equations of displacement are

$$y_{m-1q}-y_{mq}=\frac{hL_m}{(\eta S+\eta c\cos^3\gamma)b}+\frac{A_3.t}{h}z_{m-1q}+z_{mq});\quad .\quad \text{(XXIII.)}$$

$$x_{m-1q}-x_{mq}=\left\{\frac{hL'_m}{(\eta s\cos^3\beta)b}+y_{m-1q}+y_{mq}\right\}\operatorname{cotg}\beta;\quad .\quad \text{(XXIV.)}$$

and the value of Z_{mq} is

$$Z_{mq}=\frac{A_3 t}{bh^2}(L_m-L_m{}^{+1})-\frac{B_3 t^2}{h^3}(z_{m-1q}+2z_{mq}+z_{m+1q})-\frac{t^2}{h^3}.2C_2 z_{mq};$$

(XXV.)

Treating the pier as articulated at each tier

then $L_m=M_m$; $L'_m=Qh$; $Z_{mq}=0$; and we have

$$B_3 z_{m-1q}+2(B_3+C_2)z_{mq}+B_3 z_{m+1q}=A_3.\frac{h^2}{tb}(M_m+M_{m+1}) \text{ XXVI.}$$

whence z_q is obtained by successive summation, and then y_q and x_q result from XXIII. and XXIV.

Also

$$RS_{mq}+Rc_{mq}\cos\gamma=\frac{M_m}{b}; \text{ and } Rs_{mq}=Q_m.\operatorname{cosec}\beta; \quad \text{(XXVII.)}$$

The explicit approximation in this case gives, putting

$$O=\frac{A_3}{2B_3+C_2}; \qquad z_{mq}=\frac{Oh^2}{bt}(M+2Qmh+qm^2h^2). \text{ . (XXVIII.)}$$

$$\text{and } y_{m-1q}-y_{mq}=(1-OC_2).\frac{h}{b}.\frac{M_m}{\eta S} \quad . \ . \ . \ . \ . \quad \text{(XXIX.)}$$

$$RS_{mq}=(1-OC_2)\frac{M_m}{b}; \ Rc_{mq}=OC_2.\frac{M_m\sec\gamma}{b}; \ Rs_{mq}=Q_m\operatorname{cosec}\beta;$$

$$Ra'_{mq}=\frac{OC_2}{b}(M_m+M_{m+1}).\tan\gamma; \qquad Ra_{mq}=Rd_{mq}=0; \text{ (XXX.)}$$

With continuous shafts fixed at the two ends, in this case with horizontal force, reducing

$$E_1=\frac{b^2}{6r^2}-\frac{\eta S}{\eta s\cos^3\beta}+\frac{\eta S}{\eta S+\eta c\cos^3\gamma};$$

$$E_2=\frac{b^2}{6r^2}+\frac{\eta S}{\eta s\cos^3\beta}+\frac{\eta S}{\eta S+\eta c\cos^3\gamma};$$

$$E_1 H_{m-1q}+2E_2.H_{mq}+E_1 H_{m+1q}=\frac{\eta S}{\eta S+\eta c\cos^3\gamma}(M_m+M_{m+1})+qh^2\left\{\frac{b^2}{}\right.$$
$$\left.-\frac{\eta S}{\eta s\cos^3\beta}\right\}+B_3\frac{bt}{h^2}\left\{z_{m-1q}+2z_{mq}+z_{m+1q}\right\};$$

(XXXIII.)

also reducing

$$G_1=B_3t^2+6\eta Sr_1^{\,2};$$
$$G_2=(6B_3+2C_2)t^2-24\eta Sr_1^{\,2};$$
$$G_3=(10B_3+8C_2)t^2+36\eta Sr_1^{\,2};$$
$$G_1z_{m-2q}+G_2z_{m-1q}+G_3z_{mq}+G_2z_{m+1q}+G_1z_{m+2q}$$
$$=A_3\frac{h^2t}{b}\Big\{L_{m-1}+5(L_m+L_{m+1})+L_{m+2}\Big\}; \quad \ldots\ldots \quad \text{(XXXV.)}$$

Thus XXXIII. and XXXV. are the two equations affording H_q and z_q; while XXIII. and XXIV. afford y_q and x_q. The explicit approximation in this case may be treated more fully. Equation XXXIII. becomes after employing XXXV. and XXIX., neglecting the effect of q,

$$E_1H_{m-1q}+\ \&c.\ =2\{1-OC_2\}.(M+2Qmh).$$

To find an expression for H_{mq} to satisfy this, put it in form

$$H_{mq}=\frac{I}{J}(2M+4Qmh); \quad \ldots\ldots\ldots \quad \text{(XXXVI.)}$$

whence $J=\dfrac{2S}{1-OC_2}\left\{\tfrac{1}{2}b^2+\dfrac{2\eta S.r^2}{\eta S+\eta c\cos^3\gamma}\right\};$

as a first approximative value of J.

It is now necessary to find the limiting value of J through a process of successive approximation.

By substituting XXXVI. in the second member of XXXV.,

$$G_1z_{m-2q}+\&\text{c.}\ \ =12A_3h^2.\frac{t}{b}\left\{1-\frac{4I}{J}\right\}(M+2Qmh);$$

an equation satisfied by

$$z_{mq}=O.\frac{h^2}{b^2}\left\{1-\frac{4I}{J}\right\}(M+2Qmh);$$

applying this in XXXIII. the latter becomes

$$E_1 H_{m-1q} + \&c. = 2\left\{1 - O\left(C_2 + \frac{8I}{J}.B_3\right)\right\}.(M + 2Qmh)$$

whence, with XXXVI., the next closer value of J becomes

$$J = \frac{2S}{1 - O\left(C_2 + \frac{8I}{J}.B_3\right)}.\left\{\tfrac{1}{2}b^2 + \frac{2\eta S r^2}{\eta S + \eta c \cos^3\gamma}\right\}.$$

Comparing this with the former value of J, the law of formation is evident. The final value of the divisor of $2S$ will take the form

$1 - O\{C_2 + \lambda[1 - O(C_2 + \lambda(1 - \&c.))]\}$, becoming in a convergent series,

$(1 - OC_2)(1 - O\lambda).\{1 + (O\lambda)^2 + (O\lambda)^4 + \&c.$, the limit of which is $\frac{1 - OC_2}{1 + O\lambda}$; where $\lambda = \frac{8B_3 r^2}{b^2 + 4r^2(1 - 12A_3)}$.

Hence the limiting value of J is

$$\frac{2S}{1 - OC_2}.\tfrac{1}{2}b^2 + 4\eta S r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(XXXVII.)}$$

where $4\eta S r^2$ represents the resistance to flexure in the plane xy, and the other term that on the flanks to right and left $S''S''$ and $S'S'$. This reduces XXXVI. to

$$H_{mq} = \frac{2(1 - OC_2)r^2}{b^2 + 4(1 - OC_2)r^2}.(M + 2Qmh).$$

Using this to eliminate H_q from XXII., and putting

$$P = \frac{b^2}{b^2 + 4(1 - OC_2)r^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(XXXVIII.)}$$

we have $L_m = PM_m$; and $L'_m = PQ_m h$.

With the aid of this coefficient, equations XX., XXI., XXIII., XXIV., and XXXV., can be utilised, so as to transform the displacements and strains due under an articulated system into those due under continuity and fixture; the results are

$$RS_{mq}=(1-OC_2)\frac{1}{b}.PM_m\,;\quad Rs_{mq}=PQ_m\operatorname{cosec}\beta\,;$$

$$Rc_{mq}=OC_2.\frac{1}{b}.PM_m\sec\gamma;\quad Ra'_{mq}=OC_2.\frac{1}{b}.P(M_m+M_{m+1})\tan\gamma;$$

$$Ra_{mq}=Rd_{mq}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XXXIX.)}$$

Strains due to combined forces of all sorts.

RS_m or $RS'_m=\{1-T_1C_1-T_2C_2+2C_3(T_1-T_2)\}\ W_m\mp(1-OC_2).\frac{1}{b}.PM_m;$

Rs'_m or $Rs''_m=\{T_1C_1+C_3(T_1+T_2)\}\ W_m\sec\beta\mp PQ_m.\operatorname{cosec}\beta;$

Rc'_m or $Rc''_m=\{T_2C_2+C_3(T_1+T_2)\}\ W_m\sec\gamma\mp OC_2.\frac{1}{b}P_m\sec\gamma;$

Ra'_m or $Ra''_m=T_2C_2(W_m+W_{m+1})\tan\gamma\mp OC_2.\frac{1}{b}.P.(M_m+M_{m+1})\tan\gamma;$

$Ra_m=T_1C_1(W_m+W_{m+1})\tan\gamma;$

$Rd_m=C_3(T_1+T_2)(W_m+W_{m+1})\tan\beta\sec\delta;$

$=C_3(T_1+T_2)(W_m+W_{m+1})\tan\gamma\operatorname{cosec}\delta.$

Special case.—When the pier is square in plan, $t=b$; $\gamma=\beta$; $c=s$; $a=a'=a''$; $x_{mw}=z_{mw}$; $T_1=T_2$; $A_1=A_2$; $B_1=B_2$; $C_1=C_2$.

Some of the equations are simplified, and some pairs of equations coalesce.

Thus IX. and X. coalesce into

$$B_1x_{m-1w}+2(B_1+C_1+2C_3)x_{mw}+B_1x_{m+1w}=\frac{h^2}{b}.A_1(W_m+W_{m+1});$$

also XIII. and XIV. coalesce into

$E_1 x_{m-2w} + E_2 x_{m-1w} + E_3 x_{mw} + E_2 x_{m-1w} + E_1 x_{m-2w} = 12 A_1 h^2 b(W + mwh)$;
where $E_1 = B_1 b^2 + 6\eta S r^2$;

$$E_2 = (6B_1 + 2C_1 + 4C_3)b^2 - 24\eta S r^2;$$
$$E_3 = (10B_1 + 8C_1 + 16C_3)b^2 + 36\eta S r^2.$$

Also equations XVIII. coalesce into $T_1 = T_2$; and

$$x_{mw} = z_{mw} = T_1 . \frac{h^2}{b} . \tfrac{1}{2}(W_m + W_{m+1});$$

and the strains due to vertical stress are

$$RS_{mw} = \{1 - 2T_1(C_1 + 2C_3)\} W_m; \qquad Rs_{mw} = (T_1 + 2C_3) W_m \sec\beta;$$
$$Ra'_{mw} = T_1 C_1 (W_m + W_{m+1}) \tan\beta; \qquad Rd_{mw} = T_1 C_3 . 2\sqrt{2} . (W_m + W_{m+1}) \tan\beta.$$

Also as O becomes $\dfrac{A_3}{2B_3 + C_2}$; this can be substituted in the expressions for strains due to horizontal stress. Hence the strains due to combined stress can be put in simpler terms by employing these two sets of reduced values.

If the horizontal bars or the internal diagonals are entirely wanting, $a = 0$, or $d = 0$.

Special case.—If immobility is assumed, there will be no horizontal displacement under vertical stress, $x_{mw} = z_{mw} = 0$; and $T_1 = T_2 = 0$; $a = a' = \infty$; limiting value of $T_1 C_1$ is A_1; that of $T_2 C_2$ is A_2;

$$RS_{mw} = (1 - A_1 - A_2) W_m; \; Rs_{mw} = A_1 W_m \sec\beta; \; Rc_{mw} = A_2 W_m \sec\gamma.$$

Alternatively, if the internal diagonals are treated as rigid, though capable of rotation, horizontal displacement may happen, then the limit of T_1 is the negative limit of T_2;

or $\lim T_1 = -\lim T_2 = \dfrac{A_1 - A_2}{C_1 + 2(B_1 + 4B_4 + B_2) + C_2}$;

and necessarily also, $\dfrac{x_{mw}}{z_{mw}} = -\dfrac{t}{b}$.

But, when $A_1 = A_2$, that is, $\eta s \cos^3\beta = \eta c \cos^3\gamma$; there is no horizontal displacement; and this condition approximately applied to the limiting value of d indicates the section of sufficient rigidity suited to it.

Solution No. 5.—*Braced pier of two converging shafts.*

Adhering to bracing of the same type as before, this is a special case of Solution No. 3, in which one face is detached but set in a vertical plane; we may therefore employ the equations of that solution after simplification.

The symbols will be thus reduced, with a width of unity, $t=1$; c, a', d are non-existent, also ϕ, ϕ_1, ϵ_1, α, γ; and $\theta=\epsilon$; also ϕ_1 projected on the plane of the face becomes ϵ; Z_m, K_m, z_m are non-existent.

The angular relations are

$$b_m = h_m(\tan\beta_m + \tan\epsilon) = h_{m+1}(\tan\beta_{m+1} - \tan\epsilon);$$

$$\tan\beta_m = \frac{b_m + b_{m+1}}{2h_m}; \quad \sin(\beta_m - \epsilon) = \frac{b_{m-1}}{h_m} . \cos\beta_m . \cos\epsilon;$$

$$\tan\epsilon = \frac{b_m - b_{m+1}}{2h_m}; \quad \sin(\beta_m + \epsilon) = \frac{b_m}{h_m} . \cos\beta_m . \cos\epsilon;$$

$$\epsilon_m = \frac{2b_m . b_{m-1}}{b_m + b_{m-1}}; \; f_m = \frac{2b_m . b_{m-1}}{b_m - b_{m-1}}; \; g_m = \frac{b \; . \cos(\beta_m - \epsilon)}{2\sin\beta_m}.$$

The reduced coefficients are

$$A = \frac{\eta s \cos^3\beta}{\eta S \cos^3\epsilon + \eta s \cos^3\beta}; \; B = \frac{\eta S . \cos^3\epsilon . \eta s \cos^3\beta}{\eta S \cos^3\epsilon + \eta s \cos^3\beta};$$

$$C = \eta a . \left(\frac{h}{b}\right)^3; \qquad D = \frac{h_m}{h_{m+1}} = \frac{b_m}{b_{m+1}}.$$

Under vertical stress.—The general equation for vertical displacement is

$$y_{m-1w}-y_{mw}=\frac{W_m h_m}{\eta S\cos^3\epsilon_m+\eta s\cos^3\beta_m}+\left\{\frac{A_m.b_m}{h_m}-\tan\epsilon\right\}x_{m-1w}+\left\{\frac{A_m.b_{m-1}}{h_m}+\tan\epsilon\right\}.x_{mw}; \quad . \quad . \quad . \quad \text{(IV.)}$$

the fundamental relations are,

$$X_{mw}.\sec\epsilon=\frac{b_{m-1}}{h_m}A_m W_m+\frac{b_{m+1}}{h_{m+1}}.A_{m+1}W_{m+1}-\frac{b_{m-1}.b_m}{h^3_m}.B_m.x_{m-1w}-\left\{\frac{b^2_{m-1}}{h^3_m}.B_m+\frac{2b^2_m}{h^3_m}.C_m+\frac{b^2_{m+1}}{h^3_{m+1}}.B_{m+1}\right\}.x_{mw}$$

$$-\frac{b_m.b_{m+1}}{h^3_{m+1}}.B_{m+1}.x_{m+1w}; \quad . \quad . \quad . \quad \text{(VII.)}$$

With articulated tiers $X_{mw}\sec\epsilon=\frac{1}{2}(w_m+w_{m+1}).\tan\epsilon$;

and VII. becomes $(\kappa_1)_m x_{m-1w}+(\kappa_2)_m x_{mw}+(\kappa_3)_m x_{m+1w}=\frac{b_{m-1w}}{h_m}.A_m W_m+\frac{b_{m+1}}{h_{m+1}}.A_{m+1}.W_{m+1}-\frac{1}{2}(w_m+w_{m+1})\tan\epsilon$; . . (IX.)

where κ_1, κ_2, κ_3 are the coëfficients of x_w in VII.

Under the supposition that S, s, a are equal, and β constant, in every tier; A, B, C, D are constant, and suppressing the subscripts,

$$DB.x_{m-1w}+\left\{(D+D^2)B+2C\right\}x_{mw}+D^2B.x_{m+1w}=\frac{h^2_m}{b_m}\left\{A.(DW_m+W_{m+1})-\tfrac{1}{4}(1-D)(w_m+w_{m+1})\right\} \text{(IX.}a\text{)}$$

With continuous shafts fixed at the ends, the value of the series of $x_m w$ becomes

$$E_1.x_{m-2w}+E_2x_{m-1w}+E_3x_{mw}+E_4x_{m-1w}+E_5x_{m+2w}=(G_1W+G_2w_0)h^2; \quad . \quad . \quad . \quad \text{(XIII.)}$$

and $(E_1)_m$ to $(E_5)_m$ are the same as in the original Solution No. 3; substituting B for B_1, omitting B_1 and putting $\cos^3\theta=\cos^3\epsilon$; also $(G_1)_m$ $(G_2)_m$ correspond to the correlative values in No. 3,

putting A for A_1 and $\tan \epsilon$ for $\tan \phi_1$; as ϵ and ϕ_1 are on parallel planes when the face becomes vertical.

Also

$$H_{mw}=\frac{1}{6}h.X_{mw}.\sec\epsilon=\eta S\cos^3\epsilon.\frac{r^2}{h^2}\left\{x_{m-1w}+2x_{mw}+x_{m+1w}\right\} \quad \text{(XVI.)}$$

The explicit approximation, holding when $w_m=0$ and $W_m=W$; putting $T=\frac{A}{(D+D^2)B+C}$;

$$x_{mw}=T.\frac{h^2{}_m}{b_m}.\tfrac{1}{2}(1+D).W; \quad \text{(XVIII.)}$$

And the strains due to vertical stress, constant in every tier, will correspondingly be,

$$RS_w=(1-TC).W\sec\epsilon;\ Rs_w=TCW.\sec\beta;\ Ra_w=TC.2W\tan\beta; \quad \text{(XIX.)}$$

In the two first of these W_m can be substituted for W; and in the third W_m+W_{m+1} for $2W$; introducing the inherent weight w &c. of the tiers into the expression, and dealing with any one tier at a time; also the distinctive values of B, C, A, β, &c., due to each tier, with their subscripts may be introduced to obtain more exact results.

Under horizontal stress.—V'_m and V''_m remain, substituting ϵ for θ; and

$$RS_{mq}=\frac{L_m}{e_m};\ Rs_{mq}=\frac{L'_m}{f_m}; \quad \text{(XX.) (XXI.)}$$

and in the values of L_m and L'_m; $\phi_1=\epsilon$, and $\cos\epsilon_1=1$. Also the series for x_q and y_q become

$$y_{m-1q}-y_{mq}=\frac{h_mL_m}{(\eta S_m\cos^3\epsilon)e_m}-(x_{m-1q}-x_{mq})\tan\epsilon; \quad \text{(XXIII.)}$$

$$x_{m-1q}-x_{mq}=\frac{h_m L'_m}{(\eta s_m \cos^3\beta) f_m}+(y_{m-1q}+y_{mq})\operatorname{cotg}\beta_m \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{XXIV.})$$

the fundamental formulæ.

With articulated tiers. In this case, $L_m=M_m$; $L'_m=N_m$;

$$\frac{y_{m-1q}}{b_{m-1}}-\frac{y_{mq}}{b_m}=\frac{h_m M_m}{(\eta S \cos^3\epsilon)_m e^2_m}-\frac{h_m N_m}{(\eta s \cos^3\beta)_m f_m^2};$$

and the strains are inexplicit expressions in this case,

$$RS_{mq}=\frac{M_m \sec\epsilon}{e_m};\ Rs_{mq}=\frac{N_m \sec\beta_m}{f_m}; \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{XXVII.})$$

With continuous shafts fixed at the ends, an explicit direct solution is impracticable; but for the approximate explicit solution, putting $H_{mq}=\frac{I}{J}.M_m$; and $P_m=\frac{e^2_m.\cos^2\epsilon}{e^2_m.\cos^2\epsilon+4r^2_m}$;

$$\text{we have } RS_{mq}=\frac{P_m M_m}{e_m \cos\epsilon}; \qquad Rs_{mq}=\frac{P_m N_m}{f_m \cos\beta_m}; \qquad Ra_{mq}=0.$$

Proceeding to the general treatment through insoluble, but still useful, equations, that determine the relations involving H_q.

Using eq. XXIII. and XXIV., applied in Preliminary Eq. 14 of continuity and fixture, the corresponding terms are h_m is $h_m \sec\epsilon$; J is Sr^2; q is $w \sin\epsilon$; $x^n=\theta^n=0$; in the plane xy. With the shaft S'; x_m is $y'_m \sin\epsilon-x'_m \cos\epsilon$; H_m is H'_m; θ_0 is $\frac{2y_{0q}}{b_0}$. With the shaft S''; x_m is $y''_m \sin\epsilon-x''_m \cos\epsilon$; H_m is H''_m, and θ_0 is $-\frac{2y_{0q}}{b_0}$.

By subtracting the two transformed series we get

$$\left\{\frac{x_{m-1q}-x_{mq}}{h_m}-\frac{x_{mq}-x_{m+1q}}{h_{m+1}}\right\}\cos^2\epsilon-\left\{\frac{y_{m-1q}-y_{mq}}{h_m}-\frac{y_{mq}-y_{m+1q}}{h_{m+1}}\right\}.\sin\epsilon.\cos\epsilon=\frac{\cos^2\epsilon}{6\eta S\cos^3\epsilon.r^2}\times$$

$$\left\{h_m H_{m-1q}+2(h_m+h_{m+1})H_{mq}+h_{m+1}.H_{m+1q}\right\};\quad \text{(XXXI.)}$$

and through the process of Solution No. 3 adapted to this form, putting

$$(E'_1)_m=\frac{b_{m-1}.b_m}{6r^2}.\cos^2\epsilon-\frac{\eta S\cos^3\epsilon}{\eta s\cos^3\beta_m}+\left\{1-\frac{b_m}{h_m}\sin\epsilon\cos\epsilon\right\}.\left\{1+\frac{b_{m-1}}{h_m}.\sin\epsilon\cos\epsilon\right\};$$

$$(E''_1)_m=\frac{b_m.b_{m+1}}{6r^2}.\cos^2\epsilon-\frac{\eta S\cos^3\epsilon}{\eta s\cos^3\beta_{m+1}}+\text{\&c., with subscripts } m+1 \text{ instead of } m;$$

$$(E'_2)_m=\frac{b^2_m}{6r^2}.\cos^2\epsilon-\frac{\eta S\cos^3\epsilon}{\eta s\cos^3\beta_m}+\left\{1-\frac{b_m}{h_m}.\sin\epsilon\cos\epsilon\right\}^2;$$

$$(E''_2)_m=\frac{b^2_m}{6r^2}.\cos^2\epsilon-\frac{\eta S\cos^3\epsilon}{\eta s\cos^3\beta_{m+1}}+\left\{1-\frac{b_{m+1}}{h_{m+1}}.\sin\epsilon.\cos\epsilon\right\}^2;\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad \text{(XXXII.)}$$

we arrive at the general equation of the series, which holds with tiers dissimilar as regards geometric elements of bracing but with shafts of uniform section,

$$(E'_1)_m.h_m.\frac{H_{m-1q}}{b_{m-1}}+\left\{(E'_2)_m.h_m+(E''_2)_m.h_{m+1}\right\}\frac{H_{mq}}{b_m}+(E''_1)_m.h_{m+1}.\frac{H_{m+1q}}{b_{m+1}}=h_m\left\{.\frac{M_m}{e_m}.(1-\frac{b_m}{h_m}.\sin\epsilon\cos\epsilon)+\frac{N_m}{f_m}.\frac{\eta S\cos^3\epsilon}{\eta s\cos^3\beta_m}\right\}.$$

$$+h_{m+1}\left\{\frac{M_{m+1}}{e_{m+1}}(1+\frac{b_m}{h_{m+1}}.\sin\epsilon\cos\epsilon)-\frac{N_{m+1}}{f_{m+1}}.\frac{\eta S\cos^3\epsilon}{\eta s\cos^3\beta_{m+1}}\right.;\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad \text{(XXXIII.)}$$

With similar tiers and equal sections in corresponding braces, this equation is simplified, and for numerical reduction some terms may be rejected.

The last terms in the values of E are nearly 1, and may be assumed equal to 1; also in XXXIII. the terms involving N having contrary signs in the series produce small effect, and may be rejected; hence, by also rejecting the subscripts in constant terms, and putting $1-\frac{b_m}{h_m}\sin\epsilon\cos\epsilon+1+\frac{b_m}{h_{m+1}}\sin\epsilon.\cos\epsilon=2\cos^2\epsilon$ with the arithmetic mean of the terms in M,

$$(E'_1)_m.D.\frac{H_{m-1q}}{b_{m-1}}+(E'_2)_m(1+D).\frac{H_{mq}}{b_m}+(E''_1)_m.\frac{H_{m+1q}}{b_{m+1}}=\tfrac{1}{2}(1+D).\frac{M_m+M_{m+1}}{b_m}.\cos^2\epsilon.$$

With this obtaining $\frac{H_{mq}}{b_m}$; the values of $\frac{L_m}{e_m}$ and $\frac{L'_m}{f_m}$ in XXII. may be reduced, and the displacements and strains can be obtained through XXVII.

But the explicit solution will require the use of $H_{mq}=\frac{I}{J_m}.M_m$, and P_m as before given, p. 332.

Under combined stress.—The expressions for the total strains on the members are

$$RS'_m \text{ or } RS''_m=(1-T_mC_m)W_m\sec\epsilon\mp\frac{P_m}{e_m}.M_m\sec\epsilon;$$

$$Rs'_m \text{ or } Rs''_m=T_mC_m.W_m.\sec\beta_m\mp\frac{P_m}{f_m}.N_m.\sec\beta_m;$$

$$Ra_m=T_mC_m(W_m+W_{m+1}).\tan\beta_m.$$

Solution Number 6.—*Braced pier of two vertical shafts.*

Adhering to bracing of the same type as before, this is a special case of Solution No. 4, the action being limited to a single face. Employing the equations of that solution with reduced symbols, the latter are thus with a width of unity $t=1$; c, d', d are non-existent; so also Z_m, z_m, and K_m.

The reduced coefficients are

$$A=\frac{\eta s\cos^3\beta}{\eta S+\eta s\cos^3\beta}; \qquad B=\frac{\eta S\times\eta s\cos^3\beta}{\eta S+\eta s\cos^3\beta};$$

$$C=\eta a\operatorname{cotg}^3\beta; \qquad D=1.$$

Under vertical stress.—The equations of vertical displacement are, generally,

$$y_{m-1w}-y_{mw}=\frac{h}{\eta S+\eta s\cos^3\beta}\cdot W_m+\frac{b}{h}A\,.\,(x_{m-1w}+x_{mw});$$

at the top and bottom fixed sections, $x'_0=-x''_0=x'_w=-x''_w$;

$$y_{0w}-y_{1w}=\frac{h}{\eta S+\eta s\cos^3\beta}\cdot W_1+\frac{b}{h}Ax_{1w};$$

$$y_{n-1w}=\frac{h}{\eta S+\eta\cos^3\beta}\cdot W_n+\frac{b}{h}Ax_{n-1w}; \quad . \quad . \quad . \quad . \quad . \quad \text{(IV.)}$$

whence by successive summation y is given in terms of x. And the fundamental relation is

$$X_{mw}=\frac{b}{h}A(W_m+W_{m+1})-\frac{b^2}{h^3}(B.x_{m-1w}+2(B+C)x_{mw}+Bx_{m+1w}); \quad \text{(VII.)}$$

With articulated tiers, the sum $X_{mw}=0$;

$$B.x_{m-1w}+2(B+C)x_{mw}+Bx_{m+1w}=\frac{h^2}{b}.A.(W_m+W_{m+1}); \text{ (IX.)}$$

at the top and bottom sections

$$2(B+C)x_{1w}+Bx_{2w}=\frac{h^2}{b}.A(W_1+W_2);$$

$$Bx_{n-2w}+2(B+C)x_{n-1w}=\frac{h^2}{b}A(W_{n-1}+W_n).$$

With continuous shafts fixed at the ends,

$$E_1 x_{m-2w} + E_2 x_{m-1w} + E_3 x_{mw} + E_2 x_{m-1w} + E_1 x_{m+2w} = 12h^2b.A(W+mwh)\,; \quad \text{(XIII.)}$$

at the top and bottom sections

$$(2E_1 - E_2 + 2E_3)x_{1w} + (2E_2 - E_1)x_{2w} + 2E_1 x_{3w} = 2h^2b.A(9W + 11wh)\,;$$
$$2E_1 x_{n-3w} + (2E_2 - E_1)x_{n-2w} + (2E_1 - E_2 + 2E_3)x_{n-1w} = 2h^2bA\{9W + (9n - 11)wh\}\,;$$

where $E_1 = B.b^2 + 6\eta Sr^2$; $E_2 = (6B + 2C)b^2 - 24\eta Sr^2$; $E_3 = (10B + 8C)b^2 + 36\eta Sr^2$.

Whence obtaining x_w with the aid of IX., y_w will be found afterwards through IV.

Also, $H_{mw} = \frac{1}{6}hX_{mw} - \frac{r^2}{h^2}.\eta S(x_{m-1w} - 2x_{mw} + x_{m+1w})$; . (XVI.)

and at the ends

$$H_{0w} = -\tfrac{1}{12}hX_{1w} - \frac{r^2}{h^2}.\eta S(5x_{1w} - x_{2w})\,;$$

$$H_{nw} = -\tfrac{1}{12}hX_{n-1w} - \frac{r^2}{h^2}\eta S(5x_{n-1w} - x_{n-2w})\,;$$

in which X_{mw} may be replaced through VII.

With a large number of tiers, the above is inconvenient and the explicit approximative mode is preferable; in that case, putting $T = \dfrac{A}{2B+C}$;

$$x_{mw} = T.\frac{h^2}{b}.\tfrac{1}{2}(W_m + W_{m+1}) = T.\frac{h^2}{b}.(W + mwh)\,; \quad \text{(XVIII.)}$$

This equation will satisfy $n-3$ intermediate values in Eq. IX.; the series developing an inclined straight line; but if w the inherent weight were neglected the straight line would be vertical. With XVIII. Eq. IV. becomes

$$y_{m-1w} - y_{mw} = (1 - TC).\frac{h}{\eta S}.W_m\,;$$

whence the resisting section is $\dfrac{2S}{1-TC}$.

From this and from XVIII., the strains are derived

$$\left.\begin{array}{l} RS_{mw}=(1-TC)W_m\,;\ \ Rs_{mw}=TC.\,W_m\sec\beta\,; \\ Ra_{mw}=TC(W_m+W_{m+1})\tan\beta\,; \end{array}\right\} \quad . \quad . \quad \text{(XIX.)}$$

which are good approximations.

Under horizontal stress.

In this case, $V'_m=\dfrac{-H'_{m-1}+H'_m}{h}+\frac{1}{2}qh\,;\ V''_m=\dfrac{-H''_{m-1}+H''_m}{h}-\frac{1}{2}qh\,;$

$L_m=M_m-H_{m-1q}-H_{mq}\,;\qquad L'_m=Q_m h+H_{m-1q}-H_{mq}\,;$

$M_m=M+2Qh(m-\frac{1}{2})+qh^2\{m(m-1)+\frac{1}{2}\}\,;\ Q_m=Q+qh(m-\frac{1}{2});$

and the equations of displacement are

$$y_{m-1q}-y_{mq}=\frac{hLm}{b.\eta S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XXIII.)}$$

$$x_{m-1q}-x_{mq}=\left\{\frac{h.L'_m}{b.\eta s\cos^3\beta}+y_{m-1q}-y_{mq}\right\}.\cot\mathrm{g}\,\beta \quad . \quad . \quad \text{(XXIV.)}$$

at the two extremities these become

$$y_{0q}-y_{1q}=\frac{h.L_1}{b.\eta S}\,;\ \text{and}\ y_{n-1q}=\frac{h.L_n}{b.\eta S}\,;$$

$$x_{0q}-x_{1q}=\left\{\frac{h.L'_1}{b\eta s.\cos^3\beta}+y_{0q}+y_{1q}\right\}\cot\mathrm{g}\,\beta\,;\quad x_{n-1q}=\left\{\frac{hL'n}{b\eta s\cos^3\beta}+y_{n-1q}\right\}\cot\mathrm{g}\beta.$$

Treating the pier as articulated at each tier, then $L_m=M_m$; $L'_m=Q_m h$; and from summing the series XXIII. from the last to the $m+1$th; and dealing similarly with XXIV. we get

$$v_{mq}=\frac{n-m}{\eta S}\{M+(n+m)Qh+\tfrac{1}{3}qh^2(n^2+nm+m^2+\tfrac{1}{2})\}\cot\mathrm{g}\,\beta\,;$$

$$x_{mq}=\left\{\frac{n-m}{\eta s\cot\mathrm{g}^3\beta}.(Qh+\tfrac{1}{2}qh^2(n+m)\cot\mathrm{g}\beta+y_{mq}+2(y_{m+1q}+\&c.\ldots+y_{n-1q})\right\}\cot\mathrm{g}\beta.$$

Also, $RS_{mq}=\dfrac{M_m}{b}$; $Rs_{mq}=Q_m.\operatorname{cosec}\beta$; . . . . (XXVII.)

hence too $Ra_{mq}=0$, as a practical conclusion; and the relations are expressed explicitly.

Treating the shafts as continuous and fixed at the ends.

Putting $E_1=\frac{b^2}{6r^2}-\frac{\eta S}{\eta s\cos^3\beta}+1$; $E_2=\frac{b^2}{3r^2}+\frac{\eta S}{\eta s\cos^3\beta}+1$;

$$E_1H_{m-1_q}+2E_2H_{mq}+E_1H_{m+1_q}=M_m+M_{m+1}+qh^2\left\{\frac{b^2}{12r^2}-\frac{\eta S}{\eta s\cos^3\beta}\right\};$$

(XXXIII.)

the two corresponding end equations being

$$E_2H_{0_q}+E_1M_{1_q}=M_1-Q_1h.\frac{\eta S}{\eta s\cos^3\beta}+qh^2\frac{b^2}{24r^2};$$

$$E_1H_{n-1_q}+E_2H_{nq}=M_n+Q_nh.\frac{\eta S}{\eta s\cos^3\beta}+qh^2.\frac{b^2}{24r^2}.$$

With the values of H_q obtained by this series, the corresponding values of L_m and L'_m may be obtained, and applied in XXIII. and XXIV. to obtain the displacements under continuity, as well as the strains.

The explicit approximative method of satisfying XXXIII., corresponding to that in Solution 4, will be thus,

putting $H_{mq}=\frac{I}{J}(M+2Qhm+qm^2h^2)$;

the limiting value of J becomes $S\left(\frac{1}{2}b^2+2r^2\right)$

and H_{mq} becomes $\frac{2r^2}{b^2+4r^2}(M+2Qhm+qm^2h^2)$;

and putting $P=\frac{b^2}{b^2+4r^2}$; . . . . (XXXVIII.)

$L_m=PM_m$; $L'_m=PQ_mh$; whence also

$RS_{mq}=\frac{1}{b}.P.M_m$; $Rs_{mq}=P.Q_m.\mathrm{cosec}\,\beta$; $Ra_{mq}=0$; (XXXIX.)

Strains due to combined forces of all sorts.

RS'_m or $RS''_m = (1 - TC) W_m \mp \frac{1}{b} P . M_m$;

Rs'_m or $Rs''_m = TC . W_m . \sec \beta \mp PQ_m \operatorname{cosec} \beta$;

$Ra_m = TC (W_m + W_{m+1}) \tan \beta$.

Special case. If immobility be assumed under vertical stress, since $\frac{RS_{mw}}{Rs_{mw}} = \left\{ \frac{2 \eta S}{a} + \frac{\eta S}{\eta s \cos^3 \beta} \right\} . \cos \beta$;

and analytically $a = \infty$; the limit is $\frac{\eta S}{\eta s \cos^3 \beta} . \cos \beta$;

and the limit of TC is A. Hence A may be substituted for TC in the above equations of strain under the condition of immobility.

It will be noticed that in all the solutions Numbers 3, 4, 5 and 6, it has been assumed that both the braces and horizontal bars are articulated or free at the joints, and the shafts are treated alternately as articulated at each tier, and as perfectly continuous. The set of piers has been treated throughout under those circumstances.

We can now deduce relations existing, first, when the horizontal bars are fixed to the shafts and the inclined bars remain articulated; secondly, when the horizontal bars are fixed to the shafts, and there is not any inclined bracing; but these relations will be here obtained merely for the case of a pier of two vertical shafts.

Solution Number 7.—Braced pier of two vertical shafts, as in the last solution generally, but with rigidly fixed horizontal bars; the inclined braces remaining free.

Treating only of the overturning action, due to a horizontal force Q; as the horizontal displacements due to vertical stress would in most cases be inde-

finitely small, the reactions of the shafts will be discontinuous.

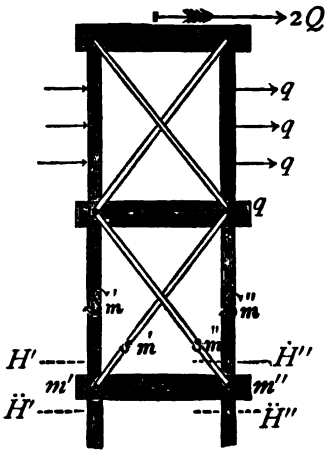


FIGURE 16.

Adding to the general notation hitherto used some symbols necessary on account of discontinuity,

let $\dot{H}'_m$ $\dot{H}''_m$ be the bending moments of the two shafts immediately above the fixtures at m', m'';

$\ddot{H}'_m$ $\ddot{H}''_m$ those below the fixtures m' and m'';

K'_m K''_m the bending moments of the fixed bar at its extremities.

Then $K'_m = \dot{H}'_m - \ddot{H}'_m$; and $K''_m = \dot{H}''_m - \ddot{H}''_m$.

Also $K_{mq} = K'_m - K''_m = \dot{H}_{mq} - \ddot{H}_{mq}$; under the method adopted throughout for purposes of distinction under horizontal stress; and $H_{mq} = \frac{1}{2}(\dot{H}_{mq} + \ddot{H}_{mq})$.

Let i and r_1 be the moment of inertia and radius of gyration of the constant section a of the fixed bar.

Adopting Preliminary Equation (1), page 291, applied the portion of shaft from m to $m+1$, we have

$$\theta_m \text{ or } \theta'_m = -\frac{h}{6\eta I}(\ddot{H}_{m-1} + 2\dot{H}_m \mp \tfrac{1}{4}qh^2) - \frac{x_{m-1} - x_m}{h}; \quad . \quad (1)$$

$$\theta_m \text{ or } \theta'_m = +\frac{h}{6\eta I}(2\ddot{H}_m + \dot{H}_{m+1} \mp \tfrac{1}{4}qh^2) - \frac{x_m - x_{m+1}}{h}; \quad . \quad (2)$$

θ'_m being positive when turning towards the right, and
θ''_m „ „ „ left.

With the shaft S', θ'_m will result from H' and x'.
„ „ S'', θ''_m „ „ H'' and x''.

Applying the same formula to the horizontal bar $m' m''$,

$$\theta'_m = +\frac{h}{6\eta i}(K''_m + 2K'_m) + \frac{y''_m - y'_m}{b}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\theta''_m = +\frac{h}{6\eta i}(K'_m + 2K''_m) - \frac{y''_m - y'_m}{b}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From these two pairs of equations, by eliminating θ'_m and θ''_m and adding the results, two basic equations are obtained in which q and $y''_m - y'_m$ disappear, connecting the terms x_{mp}, $\dot{H}_{mp}$, $\ddot{H}_{mp}$. Also from Preliminary Equation (9) we may have through similar application

$$X'_m \text{ or } X''_m = \frac{1}{h}\left\{-\ddot{H}_{m-1} + \dot{H}_m + \ddot{H}_m - \dot{H}_{m+1}\right\} \pm qh.$$

In the sum of these, which is X_{mw}, qh disappears; and this equated with Equation VII. of the last Solution affords the second typical series for the half sums of the horizontal displacements and homologous moments.

The conditions of equilibrium, both horizontally and as regards rotation at the fixed ends of the bars, are obtained in the same way as in the last Solution, and give

$$RS_{mq} = \frac{L_m}{b} = \frac{1}{b}\left\{M_m - \ddot{H}_{m-1q} - \dot{H}_{mq}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$Rs_{mq} \cdot \cos\beta = \frac{L'_m}{b} = \frac{1}{b}\left\{Q_m h + \ddot{H}_{m-1q} - \dot{H}_{mq}\right\} \quad . \quad . \quad . \quad (6)$$

Also, by the method of the last Solution expressing the strains as functions of the displacements

$$y_{m-1q} - y_{mq} = \frac{\operatorname{cotg}\beta}{\eta S} \cdot \frac{L_m}{b}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{x_{m-1q} - x_{mq}}{h} - \frac{y_{m-1q} + y_{mq}}{b} = \frac{\operatorname{cotg}\beta}{\eta s \cos^3\beta} \cdot \frac{L'_m}{b}; \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From (1) (2) (3) (4) eliminating θ'_m and θ''_m, subtracting the two equations, we obtain

$$\frac{x_{m-1q}-x_{mq}}{h}-\frac{2y_{mq}}{b}=\frac{b}{2\eta i}.K_{mq}+\frac{h}{6\eta I}\left\{\ddot{H}_{m-1q}+2\dot{H}_{mq}-\tfrac{1}{4}qh^2\right\}; \quad (9)$$

$$\frac{x_{mq}-x_{m+1q}}{h}-\frac{2y_{mq}}{b}=\frac{b}{2\eta i}.K_{mq}-\frac{h}{6\eta I}\left\{2\ddot{H}_{mq}+\dot{H}_{m+1q}-\tfrac{1}{4}qh^2\right\}; \quad (10)$$

The first members of these two equations may be eliminated; in the first case by means of the equation obtained by adding (7) and (8); in the second case by that obtained from subtracting (7) from (8) and changing m into $m+1$. Thence are derived the two typical equations connecting $\dot{H}_q \ddot{H}_q$, in the following form, as in (XXXIII.) of the last solution.

Putting $I=Sr^2$; $i=ar_1^2$; $E_1=\dfrac{b^2}{6r^2}-\dfrac{\eta S}{\eta s\cos^3\beta}+1$;

$E_2=\dfrac{b^2}{3r^2}+\dfrac{\eta S}{\eta s\cos^3\beta}+1$; adding $E_3=\tfrac{1}{2}.\dfrac{\eta Sh^2}{ar_1^2}$;

they become

$$E_1\ddot{H}_{m-1q}+(E_2+E_3)\dot{H}_{mq}-E_3\ddot{H}_{mq}=M_m+\frac{\eta S}{\eta s\cos^3\beta}\cdot Q_m h+\frac{b^2}{24r^2}\cdot qh^2;$$

$$-E_3\dot{H}_{mq}+(E_2+E_3)\ddot{H}_{mq}+E_1\dot{H}_{m+1q}=M_{m+1}-\frac{\eta S}{\eta s\cos^3\beta}\cdot Q_{m+1}h+\frac{b^2}{24r^2}\cdot qh^2; \quad (11)$$

and at the end sections they are

$$E_2\ddot{H}_{0q}+E_1\dot{H}_{1q}=M_1-\frac{\eta S}{\eta s\cos^3\beta}\cdot Q_1 h+\frac{b^2}{24r^2}\cdot qh^2;$$

$$E_1\ddot{H}_{n-1q}+E_2\dot{H}_{nq}=M_n+\frac{\eta S}{\eta s\cos^3\beta}\cdot Q_n h+\frac{b^2}{24r^2}\cdot qh^2.$$

These may be verified by Equations XXXIII. of the last solution, by supposing the fixture destroyed, and putting $\dot{H}_{mq}=\ddot{H}_{mq}=H_{mq}$. Equations (11) thus afford these sepa-

rate values, and the preceding equations will then give remaining unknown terms in displacement and strain.

Approximate explicit solution.—Treating separately the action of (Q), as composed of M the moment, and of $2Q$ and $2q$ simple horizontal forces.

With M alone all the equations of the series (11) may be satisfied by a constant value

$$\dot{H}_q = \ddot{H}_q = \frac{2r^2}{b^2+4r^2} . M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

As this coincides with the result in the last solution taken partially, it proves that the fixture of the bars has no effect on the strain produced by M.

With $2Q$ alone, temporarily rejecting $2q$, the equations (11) may be satisfied by

$$\dot{H}_{mq} = \frac{4mr^2+\lambda b^2}{4r^2+b^2} . Qh \, ; \; \ddot{H}_{mq} = \frac{4mr^2-\lambda b^2}{4r^2+b^2} . Qh \, ; \quad . \quad . \quad (13)$$

$$\text{where } \lambda = \frac{\dfrac{\eta S}{\eta s \cos^3\beta} - \frac{1}{3}}{\dfrac{b^2}{6r^2} + \dfrac{2\eta S}{\eta s \cos^3\beta} + \dfrac{\eta S . h^2}{a r_1{}^2}} \, ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Also the end equations of (11) will be satisfied by (13), remembering that at the upper end the flexible fixed horizontal bar may move, but at the lower end it cannot, those two equations taking the forms

$$(E_2 + 2E_3)\ddot{H}_{0q} + E_1 \ddot{H}_{1q} = Qh \left\{ -\frac{\eta S}{\eta s \cos^3\beta} + 1 \right\} ;$$

$$E_1 \ddot{H}_{n-1q} + (E_2 + 2E_3) H_{nq} = Qh \left\{ 2n + \frac{\eta S}{\eta s \cos^3\beta} - 1 \right\} .$$

Under the further supposition that the fixed horizontal bars at every tier are like that at the top, incapable of deformation, we should have analytically

$$r_1=\infty,\ E_3=0;\ \text{and}\ \lambda_\infty=\frac{\frac{\eta S}{\eta s\cos^3\beta}-\frac{1}{8}}{\frac{b^2}{6r^2}+\frac{2\eta S}{\eta s\cos^3\beta}};$$

then also (13) will satisfy (11); and by substitution of this value of λ_∞, the values $\ddot{H}_{0q}$ and $\dot{H}_{mq}$ can be obtained.

Hence (13) entirely satisfies the problem when verified under the condition $\lambda=0$, or $\eta s\cos^3\beta=3\eta S$.

Also applying from the former solution, page 338,

$H_{mq}=\frac{2r^2}{b^2+4r^2}(M+2Qhm+qm^2h^2)$ to the end sections, it is evidently $=0$; similarly here we have $\ddot{H}_{0q}=0$ and $K_{mq}=0$ at end sections.

Now proceeding further to utilise (13). By adding its two equations

$$\dot{H}_{mq}+\ddot{H}_{mq}=2H_{mq}=\frac{4r^2}{4r^2+b^2}\cdot 2Qmh; \quad \ldots\ldots \quad (15)$$

the effect of fixture or of its suppression is shown in the form

$$\dot{H}_{mq}\ \text{or}\ \ddot{H}_{mq}=H_{mq}\pm\frac{\lambda b^2}{4r^2+b^2}\cdot Qh.$$

Whence also, treating of the left shaft S', $\dot{H}_{mq}$ is always positive, $\ddot{H}_{mq}$ is negative when m is small or when λ is large; and H_{0q} at the top is always negative.

Subtracting equation (13) we have

$\dot{H}_{mq}-\ddot{H}_{mq}=K_{mq}=\frac{\lambda b^2}{4r^2+b^2}\cdot 2Qh$; a constant quantity at all the bars increasing with λ or with r_1 . . . . (16)

Now applying 13 to the strains (5) and (6) we get

$$RS_{mq}=\frac{b^2}{b^2+4r^2}\cdot\frac{M_m}{b}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$RS_{mq}=\frac{b^2}{b^2+4r^2}\cdot(1-2\lambda)Q\operatorname{cosec}\beta. \quad . \quad . \quad . \quad . \quad (18)$$

the former showing that the fixture of the bars does not affect the strains on the shafts; the latter that it only slightly affects the strains on the inclined braces, as λ is generally small. Also with bars of constant uniform section, of depth $2i$, we may get as maximum strain

$$Ra=\frac{K_q}{i}=\frac{b^2}{b^2+4r^2}\cdot\frac{h}{i}\cdot 2\lambda Q \quad . \quad . \quad . \quad . \quad . \quad (19)$$

in which the intensity of strain is always under ordinary conditions far less than that in the inclined brace, so that the ratio of the two may be utilised.

It may also be deduced that the fixture of the bars does not sensibly diminish the strains on the braces.

The above researches will be made use of in the next solution.

Solution Number 8.—Pier of two vertical shafts connected simply by fixed horizontal bars.

Suppressing the inclined braces of the last solution, we may deal here solely with the effects of $2Q$, for the reason that M was shown to have no effect on inclined braces.

When s and $\eta s\cos^3\beta$ tend to 0, we have from last solution

$$-\ddot{H}_{m-1q}+\dot{H}_{mq}=Qh; \text{ and } \ddot{H}_{mq}-\dot{H}_{m+1q}=-Qh;$$

that is, the shearing strains on the shafts equilibrate with $2Q$.

Transforming now the explicit approximation of the

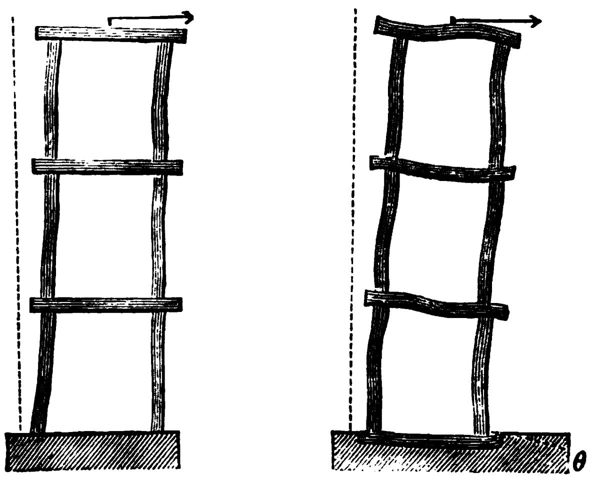
last solution, we have here $\lambda = \frac{1}{2}$; and (13) of the last solution becomes

$$\dot{H}_{mq} = \frac{4mr^2 + \frac{1}{2}b^2}{4r^2 + b^2} . Qh ; \ \ddot{H}_{mq} = \frac{4mr^2 - \frac{1}{2}b^2}{4r^2 + b^2} . Qh \quad . \ . \ . \ . \quad (1)$$

and these satisfy the above-mentioned condition.

Also $K_{mq} = \frac{b^2}{b^2 + 4r^2} . Qh$, a constant quantity. . . . . . (2)

These show that all the bending moments are independent of i, the moment of inertia of the bar section, which is approximately true, not absolutely.



Undeformed Bars. Deformed Bars.

FIGURE 17 AND 18.

Noticing that from the first two equations we get $-\ddot{H}_{m-1q} + \dot{H}_{mq} + \ddot{H}_{mq} - \dot{H}_{m+1q} = 0$; and subtracting equation (10) of the last solution from its corresponding one when in it m is changed to $m-1$, we get

$$\frac{x_{m-1q} - 2x_{mq} + x_{m+1q}}{h} - \frac{2y_{m-1q} - 2y_{mq}}{b} = \frac{b}{2\eta i} . (K_{m-1q} - K_{mq}) + \frac{h}{2\eta I} . (Qh - K_{mq}) ; \quad (3)$$

Dealing similarly with equation (9) of the last solution, but in this case changing m into $m+1$, we have

$$\frac{x_{m-1q}-2x_{mq}+x_{m+1q}}{h}-\frac{2y_{mq}-2y_{m+1q}}{b}=\frac{b}{2\eta i}(K_{mq}-K_{m+1q})-\frac{h}{2\eta I}(Qh-K_{mq}); \quad (4)$$

Also reducing equation (7) of the last solution under the conditions of this present case, we have

$$\frac{y_{m-1q}-2y_{mq}+y_{m+1q}}{b}=\frac{2\operatorname{cotg}\beta}{\eta S}\cdot\frac{Qh-K_{mq}}{b}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now, eliminating the terms in x_q by subtracting (3) and (4), and from the result eliminating the terms in y_q through (5), a general equation is obtained in a series of $n-1$ equations, connecting the $n-1$ unknown values of K_q. Adopting also the symbols E of the equation (11) of the last solution, and noticing that

$2(E_1+E_2)=\frac{b^2}{r^2}+4$; it takes the form,

$$-E_3.K_{m-1q}+2(E_1+E_2+E_3)K_{mq}-E_3K_{m+1q}=\frac{b^2}{r^2}.Qh; \quad . \quad (6)$$

and the end equations are

$$2(E_1+E_2+E_3)K_{1q}-E_3K_{2q}=\frac{b^2}{r^2}.Qh;$$

$$-E_3K_{n-2q}+2(E_1+E_2+E_3)K_{n-1q}=\frac{b^2}{r^2}.Qh.$$

These will be satisfied by substituting K_q from (2).

Besides subtracting (9) and (10) of the last solution we have $x_{m-1q}-2x_{mq}+x_{m+1q}=\frac{h^2}{2\eta I}(\dot{H}_{mq}+\ddot{H}_{mq})$ . . . . . (7)

Now adding together (3) and (4) after substituting for

K_q its value from (2); and from the result eliminating the terms in x_q through (7), we finally obtain

$$\dot{H}_{mq}+\ddot{H}_{mq}=\frac{4r^2}{4r^2+b^2}\cdot 2Qhm \quad . \quad . \quad . \quad . \quad (8)$$

a condition identical with equation (15) of the last solution, now holding with elastic fixed horizontal bars.

The analytical reduction when the horizontal bars are liable to deformation is identical with that when they are not so. In the former case the base section and the shafts themselves also turn through an angle θ_{mq}, see Figure; in the latter they do not; but the analytical results balance.

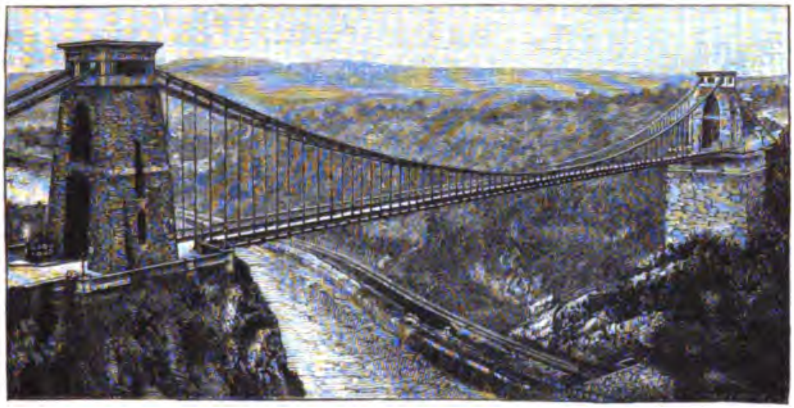
The maximum strain in the fixed section of the horizontal bar is

$$Ra=\frac{K_q}{i}=\frac{b^2}{b^2+4r^2}\cdot\frac{hQ}{i}; \quad . \quad . \quad . \quad . \quad (9)$$

The maximum strain in the shaft just above a fixed section m is

$$RS_m=RS_{mq}+\frac{2\dot{H}_{mq}}{r}=\left\{\frac{2mr(b+4r)+b(b-r)}{b^2+4r^2}\right\}\frac{hQ}{r}; \quad (10)$$

whence S and s may be determined; but this, it must be noticed, applies to these special conditions.



SECTION V.

MISCELLANEOUS SOLUTIONS.

SUSPENSION-CHAINS.—HYDRAULICS, &c.

Number 1.—General remarks on curves of suspension-chains.

Assuming that the ordinary laws of the simpler catenaries, as given in the preliminary text-books of the mathematical student, are known; it becomes necessary to discriminate among the various catenaries and curves of suspension-chains that may be practically used, before entering into a few of the more useful solutions.

The simple weightless catenary, of the abstract mathematician, certainly does not actually occur in suspension-chains of bridges; it is yet useful, in approximate preliminary calculations, as a basis to which subsequent alterations and additions may sometimes be conveniently applied.

The derived catenarian curves may afterwards be made to represent correctly the conditions of chains of *uniform section* under various modes of loading.

Parabolic curves, of a sort whose equation will be here-

after given, will represent the conditions of chains of *uniform strength* as they closely approach *catenaries of equal strength* under various modes of loading, with the implied modifications.

The curve applying to any special case can be eventually arrived at through either of these basic principles. Attempts at mixing the two, using one when the other is more applicable, or adopting approximate circular curvature, are not advisable under ordinary circumstances.

The distribution of the load on suspension-bridges being generally on pairs or sets of chains of corresponding dimensions and suited to the same curvature and spans, the load is easily divided theoretically among such chains. In practice, the lengths of the rods are adjusted by screw, and may be applied by attachment to pairs of chains through free-jointed bars.

In this country and at the present time the simpler suspension-bridges are seldom designed for anything but foot-passengers and light road-traffic in special situations. If heavier traffic has to be supported, there are two alternative usual modes of obtaining more strength and rigidity.

First, by introducing horizontal girders. The platform being stiffened throughout its length by a slightly cambered braced girder either continuous or hinged at midspan, under the most usual arrangement; this girder, or this pair of counterbraced cantilevers, relieves the chain of half the load it would otherwise bear; and the inherent weight of the girder thus introduced is an addition to the whole permanent load estimated per unit of length of span.

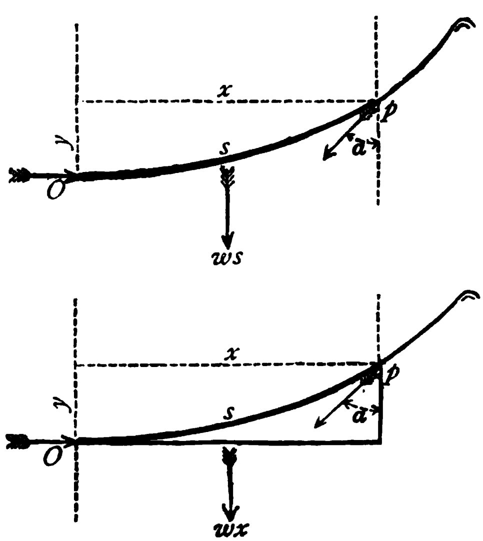
Second, by introducing stiffened or braced long link-chains. This actually amounts to substituting a deep quasi-girder for the chain, so that the additional inherent weight introduced is estimated per unit of length of curve.

The suspension-rods may be either vertical or oblique, and equidistant either along the curve or along the span.

Such are the modes of distribution most commonly adopted ; they are convenient in analytical application, and will be assumed to hold in the following solutions.

There are of course many modes of arranging the details of a suspension-bridge, and some of them do not very closely resemble any of these types ; for such cases, special solutions would be necessary.

With regard to the lofty pinnacles at each pier, sustaining the chains on saddles; this arrangement may be practically modified by introducing double brackets or cantilevers, and thus saving height; but in analytical calculations it is more convenient to treat the chain as continuous in curvature for the complete span in the first instance, the effect on the solution of breaking the curve may be afterwards introduced, if required.



FIGURES 1 and 2.—Simple equilibrated curves.

The preliminary equations of equilibrium before referred to are as follow :—

1st. For the simple catenary, with load (w) distributed uniformly per unit of length of chain, along s ; the load on Op : horizontal tension at O : oblique tension at p, as $ws : wc : w\sqrt{s^2+c^2}$; and the modulus $c = s\tan a = s.\frac{\delta x}{\delta y}$.

2nd. For the simple parabola with load (w) distributed uniformly per unit of length of span along x;

load on Op: horizontal tension at O: oblique tension at p,

as $wx : \frac{wx^2}{2y} : wx\left(1+\frac{x^2}{4y^2}\right)^{\frac{1}{2}}$;

and the parameter $m=\frac{x^2}{4y}$.

Number 2.—Suspension-chain with any vertical load proportioned to the length of chain.

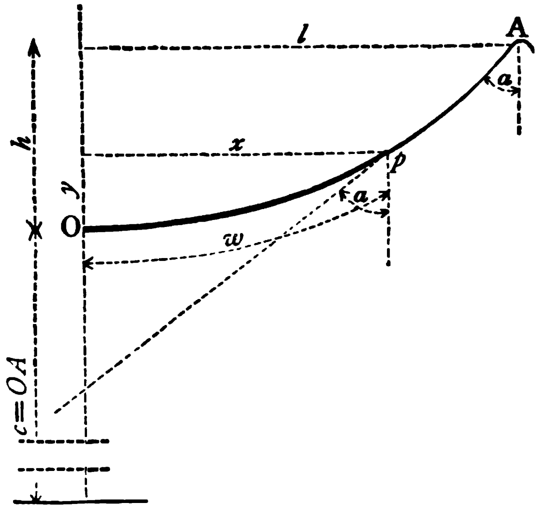


FIGURE 3.

Let x and y be the co-ordinates from the lowest point O of any point p in the chain;

let $\frac{1+m}{1}$ be the ratio of increment of weight of chain and its attached load;

c the tension at the lowest point;

w the weight of chain and load from the lowest point up to the point p;

The tension at p is then $(1+m)(c+y)$;

but as $\frac{\partial x}{\partial y}=\frac{c}{w}$; and $\frac{\partial x}{\partial y}=\frac{w}{(c^2+w^2)^{\frac{1}{2}}}$ is the ratio of the weight to the tension at p; then $y+c$ is the tension at p.

Now as $\frac{\partial y}{\partial x}=\frac{w}{c}$; $\partial_x^2 y+1=\frac{1}{c^2}(w^2+c^2)=\partial_x^2 w$;

whence $\frac{1}{c}.\partial_w x=\frac{1}{(w^2+c^2)^{\frac{1}{2}}}$; integrating with respect to w, and noticing that when $x=0$, $y=0$;

$$\frac{1}{c}.x=\log.\frac{w+(w^2+c^2)^{\frac{1}{2}}}{c};$$

$$\therefore\ \epsilon^{\frac{x}{c}}=\frac{1}{c}\left\{w+(w^2+c^2)^{\frac{1}{2}}\right\};\ \text{and}\ \epsilon^{-\frac{x}{c}}=\frac{1}{c}\left\{(w^2+c^2)^{\frac{1}{2}}-w\right\};$$

$$\text{hence}\ w=\tfrac{1}{2}c\left(\epsilon^{\frac{x}{c}}-\epsilon^{-\frac{x}{c}}\right);$$

and by substitution $\frac{\partial y}{\partial x}=\frac{w}{c}=\frac{1}{2}\left(\epsilon^{\frac{x}{c}}-\epsilon^{-\frac{x}{c}}\right)$;

integrating with respect to x and noticing that when $x=0$, $y=0$, we have the equation to the curve

$$y+c=\tfrac{1}{2}c(\epsilon^{\frac{x}{c}}+\epsilon^{-\frac{x}{c}});$$

whence also $x=c.\log.\frac{w+(w^2+c^2)^{\frac{1}{2}}}{c}$.

Having thus determined the catenary, the remaining relations that may be required are thus, since w is proportionate to s the arc up to p,

$$s=\tfrac{1}{2}c(\epsilon^{\frac{x}{c}}-\epsilon^{-\frac{x}{c}})=c.\partial_x y=(y^2-c^2)^{\frac{1}{2}};$$

when α is the inclination to verticality of the chain at p, and r is the radius of curvature at p,

$$\text{cotg}\,\alpha=\partial_x y=\frac{s}{c}=\tfrac{1}{2}(\epsilon^{\frac{x}{c}}-\epsilon^{-\frac{x}{c}});\ \text{and}\ r=\frac{y^2}{c}=\tfrac{1}{4}c(\epsilon^{\frac{2x}{c}}+\epsilon^{-\frac{2x}{c}}+2);$$

and the tension at p is $y+c=\frac{1}{2}c(\epsilon^{\frac{x}{c}}+\epsilon^{-\frac{x}{c}})$;

and the area between the curve and the directrix is

$$\int_x y = cs = \tfrac{1}{2}c^2(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}).$$

Also the vertical pressure on the saddle at the top of a pier will be the whole weight between the lowest points of the two adjoining spans.

In reduction, the term $\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}} = n$, being the difference between the Naperian antilogarithm of $\frac{x}{c}$ and the Naperian antilogarithm of its reciprocal, can be obtained for various assumed values of c, and for some fixed convenient value of x, as 1000; the tensions, weights, and arcs can then be tabulated or worked by diagram and scale, or, as it is termed, the graphic method; the whole can then be applied by proportion to any required value of x representing the half-span.

With common logarithms, &c., put $\epsilon^{\frac{x}{c}} = 10^{\cdot 4343\frac{x}{c}}$; as Brigg's modulus is 0·434294; the common logarithm of this modulus is $\bar{1}\cdot 637784$. (For short Naperian logarithms and tables for conversion of logarithms, see 'Accented Four-Figure Logarithms,' Allen, 1881.)

It will be noticed that c is the parameter of the curve (properly $= OA$ but cut through in the Figure) and y the ordinate to any point p is the tension at that point; both representing weights of chain equal to such tensions; while s the length of arc up to p is w the weight supported between p and the lowest point on the curve. The terms used thus strictly correspond.

The value of c, if required, may be obtained by successive approximation through the foregoing equations, the given quantities then being s' the length of chain under consideration, and two values of x and two of y, as co-ordinates of its two extremities.

The maximum value in the series of calculated tensions, which must be less than the permissible extreme tension, may be equated with $(c+y)(1+m)$ to obtain the greatest value of y, the depth from the point of suspension that the conditions of the case admit; or if the series of values be tabulated, the required values of c and of y may be obtained through them direct.

The conditions of this solution may as before mentioned serve for some preliminary purposes; the solution itself is in England usually assigned to Whewell.

Number 3.—Suspension chain loaded in any way.

While giving preliminary equations that may be useful after modification suited to any particular case, a general representative equation applicable to various modes of loading may be mentioned.

Let x and y be the co-ordinates of any point p in the chain from the origin O at the lowest point;

W the total weight of chain, rods, and roadway from O to p;

w_1 the weight of the chain per unit of arc s;

w_2 the weight of the roadway per unit of its length;

K the weight of the rods per superficial unit of a representative vertical surface, that expresses them as uniformly distributed.

Then $W=c.\frac{\partial y}{\partial x}$, according to preliminary principles, if the curve is catenarian of any sort.

Also with reference to a small ordinate ∂x,

$w_1\partial s$ = weight of its corresponding portion of curved chain;
$w_2\partial x$ = weight of its corresponding portion of roadway;
$Ry\partial x$ = weight of its corresponding portion of the rods;

Hence the whole weight corresponding to an elementary portion ∂W is $w_1 \partial s + w_2 \partial x + R y \partial x$;
and the differential coefficient of the whole weight is $w_1 + w_2 . \frac{\partial x}{\partial s} + R y . \frac{\partial x}{\partial s}$; hence the whole weight W is the general integral of this expression commencing from the origin, when $x=0, y=0$;

or $w_1 s + w_2 x + R . \int y . \frac{\partial x}{\partial s} = c . \frac{\partial y}{\partial x}$;

the general equation originally due to Hodgkinson.

In special applications of the catenary, the whole load, both rods and roadway, may correspondingly be represented by a theoretical vertical surface lying between the directrix and the length of curve under consideration.

In the last, Number (2) Solution, where the load is proportioned to the length, the area would be $cs = \frac{1}{2} c^2 (\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}})$; thus limiting the case to one where c is the ordinary modulus of the curve. But under altered conditions, the load is still equal to an intercepted area, if the load be proportioned to the vertical ordinate; but then c_1 is a new value replacing c (see Figure); the properties of the upper catenary Op are retained in the lower catenary $O_1 p_1$, with the exception that the vertical forces are all altered in the ratio of c_1 to c.

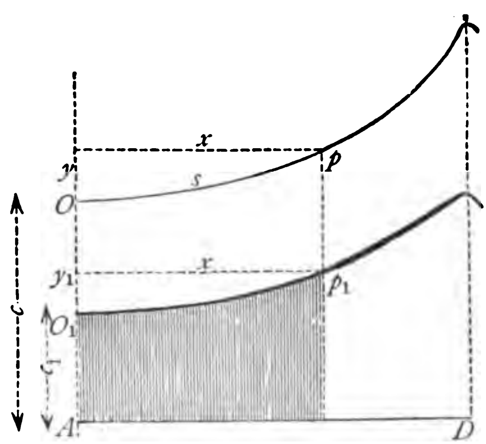


FIGURE 4.

Hence the horizontal tension at O_1 is the same as that at O, or corresponds; but the load from O to p becomes

the load from O_1 to p_1, and the tension at p becomes the tension at p_1; these two latter forces being changed by merely applying a factor $\frac{c_1}{c}$. The values given in the last Solution No. 2 can therefore be easily modified to suit the altered conditions.

Also, if the value of c be required from given conditions of any such modified catenary; let c_1 be given, and x and y the co-ordinates of p_1;

then $x = c.\log\left\{\left[y_1 + (y_1^2 - c_1^2)^{\frac{1}{2}}\right].\frac{1}{c_1}\right\}$;

an equation from which c may be obtained.

Besides, if the rods and loading be oblique, as well as proportioned to the vertical ordinates, the further modifications due to oblique loading may be applied in this case also in a manner similar to that before shown in Solution No. 2; where the load is proportioned to length.

Number 4.—Suspension chain of uniform strength, neglecting the weight of the rods.

In this case the section of the chain is everywhere proportioned to the tension.

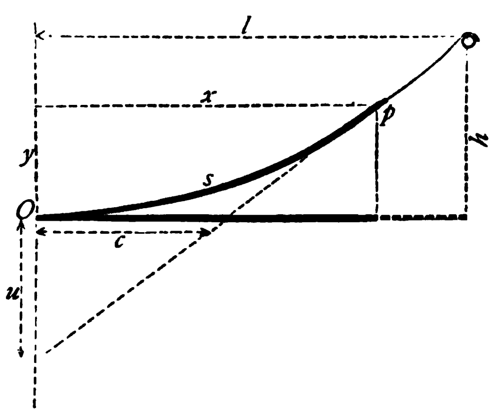


FIGURE 5.

Let x and y be the co-ordinates of the point p,

s the length of arc from the origin O to p,

A the aggregate section of the chain at p,

w_1 the weight of a linear foot of the chain,
w_2 the weight of a linear foot of the roadway.
c the tension at O;
u the whole load on Op;

Then $u = w_1 \int A . \partial s + w_2 . x$ . . . . (I.)

Also graphically by resolution of forces u and c we shall evidently have $\frac{\partial y}{\partial x} = \frac{u}{c}$; and the strain at p will be $(c^2 + u^2)^{\frac{1}{2}}$.

Putting R = resistance of the chain to tension per unit of sectional area, and supposing that the uniformity of strength throughout the chain is analytically represented by putting its strength = m times the strain on it everywhere, we have

$$AR = m(c^2 + u^2)^{\frac{1}{2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{II.})$$

$$\therefore AR = mc\left(1 + \frac{u^2}{c^2}\right)^{\frac{1}{2}} = mc\left(1 + \partial^2{}_x y\right)^{\frac{1}{2}} = mc . \partial_x s;$$

$$\text{or} . \partial_x s = \frac{AR}{mc}.$$

$$\text{Also} \int A . \partial s = \int A . \frac{\partial s}{\partial x} . \partial x = \frac{R}{mc} \int A^2 . \partial x = \frac{m}{Rc} . \int_x (c^2 + u^2);$$

$$\therefore u = \frac{mw_1}{Rc} . \int_x (c^2 + u^2) + w_2 x.$$

Differentiating with respect to x, and observing that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} . \frac{\partial y}{\partial x} = \frac{u}{c} . \frac{\partial u}{\partial y};$$

$$\therefore \partial_x u = \frac{u}{c} . \partial_y u - \frac{mw_1}{Rc} . (c^2 + u^2) + w_2 = \frac{mw_1}{Rc}\left(u^2 + c^2 + \frac{Rcw_2}{mw_1}\right);$$

$$\therefore x = \frac{Rc}{mw_1} . \int_0^u \frac{\partial u}{u^2 + c^2 + \frac{Rcw_2}{mw_1}}; \quad y = \frac{R}{mw_1} \int_0^u \frac{u \partial u}{u^2 + c^2 + \frac{Rcw_2}{mw_1}};$$

whence by integration

$$x = \frac{Rc}{mw_1}\left(c^2 + \frac{Rcw_2}{mw_1}\right)^{-\frac{1}{2}} . \tan^{-1}\left(c^2 + \frac{Rcw_2}{mw_1}\right)^{-\frac{1}{2}} . u;$$

$$y = \frac{R}{2mw_1} \cdot \log_e \left\{ \frac{u^2 + c^2 + \dfrac{Rcw_2}{mw_1}}{c^2 + \dfrac{Rcw_2}{mw_1}} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III.)}$$

And by substitution in the value of u, and reduction,

$$y = \frac{R}{mw_1} . \log \sec \left\{ \frac{mw_1}{R} \left[1 + \frac{Rw_2}{cmw_1} \right]^{\frac{1}{2}} . x \right\} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV.)}$$

the equation to the curve of uniform strength. By substituting in (II.) the value of u from III.,

$$A = \frac{mc}{R} . \left\{ 1 + \left(\frac{1 + Rw_2}{mcw_1} \right) . \tan^2 . \frac{mw_1}{R} \left(1 + \frac{tw_2}{mcw_1} \right) x \right\} ;$$

showing that the section increases from the lowest point to the highest, where it is a maximum.

The weight of the chain is from Equation (I.) $w_1 \int A . \delta s = u - w_2 x$; and substituting in this the value of u from Equation III., it becomes

$$c \left(1 + \frac{Rw_2}{mcw_1} \right)^{\frac{1}{2}} . \tan \left\{ \frac{mw_1}{R} \left(1 + \frac{Rw_2}{cmw_1} \right)^{\frac{1}{2}} . x \right\} - w_2 x .$$

To obtain the value of c, the tension at the lowest point ; let $2l$ = the whole span,

h = height from lowest to highest point vertically,

then h and l being extreme values of y and x in Equation IV., we have

$$h = \frac{R}{mw_1} . \log \sec \left\{ \frac{mw_1}{R} \left(1 + \frac{Rw_2}{cmw_1} \right)^{\frac{1}{2}} . l \right\} ;$$

and solving this with regard to c,

$$c = \frac{Rw_2}{mw_1} \left\{ \left(\frac{R}{mw_1 l} . \sec^{-1} \epsilon^{\frac{mw_1}{R} . h} \right)^2 - 1 \right\}^{-1} .$$

Number 5.—*Chain of uniform strength, allowing for weight of vertical rods.*

Adopting the general notation of the last Solution, No. 4; and an amplification of its method;

let w_3 be the weight per square foot of a lamina representing the weight of the rods (see also Number 3 for this principle).

Then in this case, the whole weight on a given portion of chain from O to p will be

$$u=w_1\int A\,.\,\partial s+w_2 x+w_3\int y\partial x; \quad . \quad . \quad . \quad \text{(I.)}$$

as in the preceding solution

$$\frac{\partial y}{\partial x}=\frac{u}{c}; \quad AR=m(c^2+u^2)^{\frac{1}{2}}; \quad \int A\,.\,\partial s=\frac{m}{Rc}\,.\int(c^2+u^2)\,.\,\partial x; \text{(II.)}$$

substituting in equation (I.), differentiating with respect to x, and observing that $\partial_x u=\frac{u}{c}\,.\,\partial_y u$;

$$\partial_x u=\frac{mw_1}{Rc}\,.\,(c^2+u^2)+w_2+w_3 y; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III.)}$$

Putting $a=\frac{mw_1}{R}$; and b=length of shortest rod, by transposing and reducing the last,

$$\partial_y{}^2 u-2au^2=2c(w_3 y+ac+w_2)$$

This linear equation in u^2, when integrated, will give

$$u^2\epsilon^{-2ay}=2c\int(w_3 y+ac+w_2)\epsilon^{-2ay}\,.\,\partial y+C$$

which when reduced by integrating between the limits b and y; observing that when $y=b$, $u=0$; becomes

$$u^2\epsilon^{-2ay}=\frac{c}{a}\left\{w_3(b\epsilon^{-2ab}-y\epsilon^{-2ay})+\left(\frac{w_3}{2a}+ac+w_2\right)(\epsilon^{-2a}-\epsilon^{-2ay}\right\};$$

$$\therefore u^2=\frac{c}{a}\left\{w_3(b\epsilon^{-2a(y-b)}-y)+\left(\frac{w_3}{2a}+ac+w_2\right)(\epsilon^{2a(y-b)}-1)\right\}; \quad \text{(IV.)}$$

Substituting this value in Eq. (II.), and reducing

$$A=\frac{(ac)^{\frac{1}{2}}}{w_1}\left\{\left(\frac{w_3}{2a}+w_3b+ac+w_2\right)\epsilon^{2a(y-b)}-w_3y-\frac{w_3}{2a}-w_2\right\}^{\frac{1}{2}}; \quad \text{(V.)}$$

thus the variation in section of the chain of uniform strength is found for this case.

The process that would arrive at the equation to the curve, in a rather complicated form, is thus: Differentiating the equation $\partial_x y=\frac{u}{c}$, with respect to x, and substituting for $\partial_x u$ its value from eq. (III.);

$$c.\partial_x^2 y=\frac{a}{c}.(c^2+u^2)+w_2+w_3y;$$

substituting for u^2 its value from Equation (IV.)

$$c.\partial_x^2 y=\left(\frac{w_3}{2a}+w_3b+ac+w_2\right)\epsilon^{2a(y-b)}-\frac{w_3}{2a};$$

multiplying both sides of this by $\partial_x y$, integrating between the limits b and y, observing that when $y=b$, $\partial_x y=0$, we have

$$ac.\partial_x{}^2 y=\left(\frac{w_3}{2a}+w_3b+ac+w_2\right)(\epsilon^{2a(y-b)}-1)-w_3(y-b); \quad \text{(VI.)}$$

In order to proceed further with useful analytical results, it becomes necessary to approximate by rejecting terms, and transforming the catenary into a parabola.

Parabolic Approximation.

As $a=\frac{mw_1}{R}$; and usually R is very large compared with m and w_1; a is very small; and all terms of the series $\epsilon^{2a(y-b)}$ that involve powers of $2a(y-b)$ above the first may be rejected; hence $\epsilon^{2a(y-b)}-1=2a(y-b)$.

Equat. (VI.) then becomes $c.\partial^2{}_x y=2(w_3b+ac+w_2)(y-b)$.

Taking the square root of both sides, transposing, and integrating

$$x^2 = \left(\frac{2c}{w_3 b + ac + w_2}\right) \cdot (y - b) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VII.)}$$

the required parabola suited to the case.

Substituting for x and y their extreme values l and h, the half-span and rise, and solving

$$c = l^2 \left\{ \frac{w_2 + w_3 b}{2h - 2b - al^2} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VIII.)}$$

Thus the tension at the lowest point is known. So that by substituting this in Equation VII. the length of suspension rod y at any distance x may be found.

As we have already, from the rigidly correct solution, a general value of A in Equation (V.) giving the section anywhere, we may now obtain from it the particular value of A_1 at the lowest point of the chain, by substituting the approximate terms for the others, and reducing, so that

$$A_1 = \frac{ac}{w_1} \left\{ 2(y - b)\left(\frac{w_3 b + w_2}{c} + a\right) + 1 \right\}^{\frac{1}{2}}.$$

It may be noticed that the general parabola for approximation to the general catenary in preliminary investigations has a focal distance equal to half the modulus c of the catenary of uniform section; the lengths of arc, the vertical ordinates, and the included areas, closely approximate, but the inclinations of the curve do not; these last should be avoided, when using the one curve to represent the other.

Number 6.—Chain with oblique suspension rods.

This mode of construction produces greater stiffness, its analytical application is simply introduced as a modification into the results of the preceding cases (see Remarks in Number 3).

When the loading is proportionate to the length of chain, we can apply the necessary modification to the results of Solution No. 2.

Let β be the inclination to verticality of the oblique rods x, and y being the co-ordinates of p with vertical rods x' and y' are the new co-ordinates introduced with oblique rods; and α' is the new inclination to verticality of the tangent at p, instead of α. So that $y' = y \sec \beta$; and the tangent is now $= y' \cos \beta \sec \alpha'$.

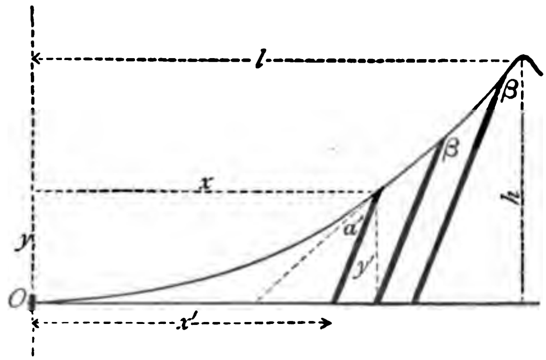


FIGURE 6.

Also if W be the total load vertically on Op, W' the oblique load, $= W \sec \beta$; the horizontal tension at O will be altered by the ratio $\frac{x'}{x}$; and the tension at p may then be obtained from the conditions of equilibrium, whatever they may be, in any case.

The three equilibrated forces acting on p are the oblique load up to p, the horizontal tension at O, and the tension at p; these are evidently in the ratio

$$y' : y . \frac{\sin(\alpha' - \beta)}{\cos \alpha'} : y' \frac{\cos \beta}{\cos \alpha};$$

and if w be the vertical load per unit of length of span, or

of platform, the oblique load becomes $wx . \sec\beta$; and the above ratio becomes

$$wx\sec\beta : wx\,(\tan\beta - \tan\alpha') : wx\sec\alpha'$$

in which the values of the three forces are now correctly given.

The necessary transformation from conditions of vertical loading can now be effected through these values.

The solutions of the catenary of equal strength, with and without taking the weight of the rods into account, are those of Moseley; to whom also the general elucidation of the subject is principally due.

HYDRAULICS.

Number 1.—*The curved dam.*

The effect of curving a dam in plan is apparently to transmit some horizontal pressure to the flanks, and thus to reduce the direct pressure normal to the curve of the dam; it is hence necessary to determine the economy of section resulting from any such curvature. The occasion, when this is of interest, occurs when a narrow gorge having sides of firm rock is the site of some proposed dam.

Should there be any economy from curvature, it will affect the section in the following way.

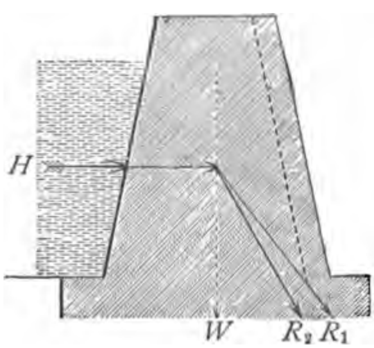


FIGURE 7.

The section in the figure is acted on by two forces, the horizontal pressure H representing concentrated fluid pressure, acting at the centre of gravity of the section, and

the vertical weight of the section W acting at the same point; their resultant would with a dam, straight in plan, take the direction R_1. But with a curved dam, if any economy is effected, the resultant will take some direction R_2, putting the base at a greater distance from the outer foot. Hence, if the same limit of stability be preserved, the effect of curving the dam is to allow a slice, roughly shown in Figure to be economised in the section, without any detriment to stability.

The solution of the question involves a limit of stability, or modulus, a limiting pressure, and the determination of the thickness of dam when curved. The problem slightly resembles that of determining the thickness of an arch under uniformly distributed vertical load, but it is simpler as the pressures are everywhere normal instead of parallel, and are equal instead of varying with the depth of load line.

Assuming that the curvature is circular, and symmetrical with reference to the general axis of the gorge, one-half of the dam alone requires treatment. Also a limiting distance or modulus of one-third the thickness at the base may be assumed, while at this ultimate position the thrust must not exceed some given limiting pressure. Equilibrium will also require that the curve of pressures in plan be everywhere at right angles to the direction of fluid-pressure, that is, the pressure-curve is concentric with the curvature of the dam.

Taking a horizontal section of the half-dam in the Figure,

Let A represent the reaction of the other half-dam,

B any resistance afforded at the flank,

P the fluid pressure on the arc from the flank as far as any point n.

Then at any section, located at n, equilibrium will exist between A, B and P; and taking the point b as a point of reference for moments; a, b, p, the leverages of the three forces with regard to it; $b=0$, and $Aa=\Sigma Pp$ (I.)

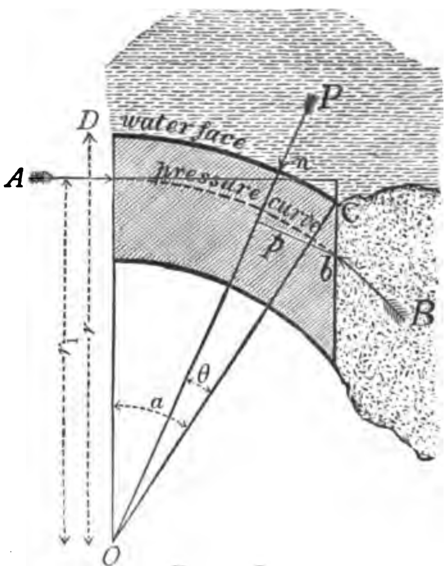


FIGURE 8.

To obtain directly comparable instead of mere representative terms, in this equation,

Let α be the angle subtended by the half-dam at the centre of curvature,

Let r the radius of curvature of the water face,

r_1 the radius of curvature of the pressure curve,

w the water pressure per unit of surface,

δs an elementary portion of that surface,

θ the variable angle formed at the centre of curvature by any section n with the line OC;

Now a evidently$=r_1(1-\cos\alpha)$;

$\delta P=w.\delta s=wr\delta\theta$; and $p=r_1\sin\theta$;

also $\Sigma Pp = {}_0^{\alpha}\!\int_{\theta} .wr.r_1 \sin\theta = wrr_1.(1-\cos\alpha)$.

So that equation (1) takes the simple form,

$A=wr$. Also we may now treat w, the water pressure per unit of surface, with regard to profile; supposing the horizontal section to be taken at a depth h below the summit of the dam, and δ to be the density of the fluid; we have $w=\frac{1}{2}h^2\delta$,

hence $A=\frac{1}{2}h^2\delta.r$ . . . . (II.)

Having determined thus the reaction of the other half-dam, we may now find x the thickness necessary for the dam.

The condition, that the pressure per unit of surface at the section treated, and at the point in it where it is greatest, shall be equal to the limiting pressure, may be thus expressed

$$\tfrac{2}{3} \cdot \frac{P}{q\delta_1} = H;$$

where q=distance of the pressure line from the nearest edge of the section,

δ_1=density of the masonry,

H=the extreme height possible in the vertical wall, that will not cause the limiting pressure A to be exceeded.

Putting $q=\frac{1}{3}x$; $P=A$, we have $2A=xH.\delta_1$; and by substituting in Equation (II.)

$$x=\frac{hr^2}{H} \cdot \frac{\delta}{\delta_1} \quad \text{. . . . . (III.)}$$

This equation does not yield directly any definite result by itself. Noticing that $\frac{x}{h}$ the ratio of thickness to height is the required quantity; we necessarily look for some hypothesis connecting x with r the radius of curvature. In the theory of the arch there is no precise law connecting the radius with the ring-thickness, so that its principles will not aid in this problem.

Assuming now that the hypothesis that led to Equation (III.) will fail when the thickness exceeds one-third of the radius, we have at the limit, when $x=\frac{1}{3}r$;

$$h=\left(\tfrac{1}{3} H . \frac{\delta_1}{\delta}\right)^{\frac{1}{2}}.$$

Reducing this to actual conditions by adopting ordinary numerical values, let $\delta_1 = 2\delta$; and let A the limiting pressure possible for the masonry used be 200 talents (footweight) per square foot; the value of h resulting will be about 15 feet. With other practicable figures it might amount to 20 feet.

Hence it may be concluded that with dams exceeding 20 feet in height the thickness would be great in proportion to the radius of curvature, to allow of any economy resulting from curvature and presumed thrust on the flanks.

That is to say, that any substantial economy of section can only be effected in low dams, not exceeding 15 feet in height; in which the actual economy would be necessarily small in amount and in real value.

This solution is originally due to Delocre, but has been modified.

Number 2.—The curved lock-gate.

Barlow's solution of the most favourable arrangement and best curve for a pair of timber lock-gates of small size is practically correct under the limited conditions for which it was intended.

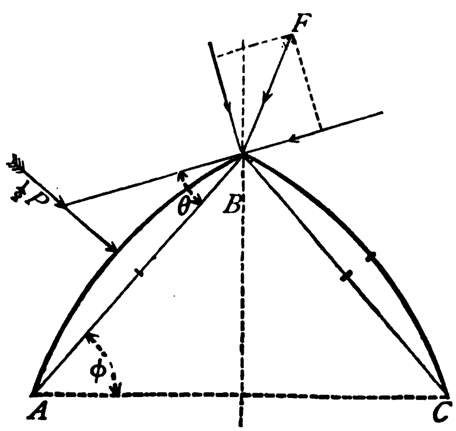


FIGURE 9.

It is not, however, strictly correct as a general deduction, nor is it capable of more extended application.

Let AB, BC be the pair of lock-gates, AC an axis across the channel. See Figure.

ϕ, the inclination of the chord of AB to the axis AC;

θ, the inclination of the chord BA to the tangent of the curve at B;

P, the total pressure on a single lock-gate;

F, the force acting at B at right angles to the curvature of the other gate BC.

Then the equivalent transverse pressure on the gate AB concentrated at its middle will be $\frac{1}{2}P$: the tangential stress will be $\frac{1}{2}P$ cosec θ; and the reaction F is evidently $\frac{1}{2}P$, on this partial hypothesis.

Resolving F tangentially and normally to the curvature of the gate AB, the latter stress is supposed not to affect it, but the tangential stress exerts compression, and is equal to $\frac{1}{2}P.\text{cosec}\,(\phi-\theta)$. This will be partly counteracted by the before-determined tangential stress, whose value was $\frac{1}{2}P.\text{cosec}\,\theta$.

Now, it is required to obtain a representative transverse stress Q acting at the middle of the lock-gate AB, which will comprise the effect of the whole of the stresses on it, and which can be substituted for $\frac{1}{2}P$, the real transverse stress, that was independent of them.

$$\text{Then}\ \frac{Q}{\frac{1}{2}P}=\frac{\frac{1}{2}P.(\text{cosec}\,\theta-\text{cosec}\,\overline{2\phi-\theta})}{\frac{1}{2}P.\text{cosec}\,\theta};$$

$$\text{or}\ Q=\frac{1}{2}P.\left\{1-\frac{\sin\theta}{\sin(2\phi-\theta)}\right\}.$$

From this it is evident that Q will be a minimum when $\phi=\theta$; that is, when the curvature is continuous and circular.

As it would be difficult to maintain permanently a perfectly continuous curve; owing to wear of heelposts and set of masonry, a slight salient of 1 or 2 feet is usually given in locks 40 to 60 feet wide. With given values of

the radius and of ϕ and θ, the values of Q may be numerically determined, and the dimensions of material based on the single representative stress Q.

Number 3.—Clear overfall or weir.

Let the overfall be rectangular occupying the whole width of the channel; its dimensions must admit of constant discharge throughout the channel.

Let Q be the given discharge.
- h_1 the head due to velocity of approach.
- h_2 the required depth of water in the channel of approach.
- V_1 the velocity of approach.
- V the velocity at the overfall.
- o the velocity coefficient for the overfall.
- b the breadth of overfall.

The lower water surface will stand at a depth x below the weir-sill, varying from h_2 to zero; and we shall have

$$\delta_x Q = o \times bV.\delta x = o.b.\delta z\sqrt{V_1^2 + 2gx};$$

$$\therefore Q = o.b\sqrt{2g}\,{}_0\!\int_x^h \sqrt{\frac{V_1^2}{2g} + x} = o.b\sqrt{2g}\,{}_0\!\int_x^h (h_1 + x)^{\frac{1}{2}};$$

$$\text{or } Q = o \times \tfrac{2}{3} b\sqrt{2g}\left[(h_1 + h_2)^{\frac{3}{2}} - h_1^{\frac{3}{2}}\right].$$

$$\text{whence } h_2 = \left\{\frac{3Q}{o.b.\sqrt{2g}} + h_1^{\frac{3}{2}}\right\}^{\frac{2}{3}} - h_1.$$

Hence h_2 can be determined when Q, h_1, are given; Q being obtained from gauging. But if b be unknown as well as h_2, successive approximation must be applied to the formula.

Number 4.—Drowned overfall or submerged weir.

Let the form be rectangular, required the dimensions, so that the discharge of the channel may be constant.

Let Q be the given discharge of the channel,

- q_1 and q_2 the discharges of the drowned and free sections,
- b the breadth of the river,
- h_1 the height due to velocity of approach,
- h_2 the height of the upper water surface above the lower water surface,
- h_3 the height of the lower water surface above the drowned sill.

Then will $q_1 = o \times bh_3\{2g(h_1+h_2)\}^{\frac{1}{2}}$;

$$q_2 = o \times \tfrac{2}{3} b \sqrt{2g}\{(h_1+h_2)^{\frac{3}{2}} - h_1^{\frac{3}{2}}\};$$

$$\therefore h_3 = \frac{Q}{o \times b.\{2g(h_1+h_2)\}^{\frac{1}{2}}} - \tfrac{2}{3}\left\{h_1 + h_2 - \left(\frac{h_1^3}{h_1+h_2}\right)^{\frac{1}{2}}\right\}.$$

From this either h_3 or b may be severally obtained, or both of them under successive approximation.

Also if x be the depth of water in the lower channel, V_1 and V_2 the velocities of approach and of departure,

$$\text{then } Q = V_1 b(h_2 + x) = V_2 . bx;$$

$$\text{and } h_1 = \frac{V_2^2}{2g}\left(\frac{x}{h+x}\right)^2;$$

whence x may be obtained.

Number 5.—Water-pipes.

Thickness of water-pipes and cylinders to resist internal pressure. Although theory still fails to assign precise thickness due to given pressure and strength of material, the

various attempts to deduce a law for it have been partly utilised in the empirical formulæ usually employed, and hence possess some interest.

One of the oldest is that of Barlow. According to him, the resistance under pressure is proportional to the extension of material divided by the length affected, that is, the circumference of each elementary annulus; and this resistance is greater at the exterior than at the interior circumference. But the area of the whole ring of metal is supposed to be entirely unaffected in value, when strained. His formulæ are thus derived,

let D and D_1 be the original interior and exterior diameters,

$D+d_1$ and D_1+d_1 the stretched interior and exterior diameters,

R and R_1 the interior and exterior resistances.

As the area is proportional to the squares of the diameters,

$$D_1^2-D^2=(D_1+d_1)^2-(D+d)^2;$$

and $\dfrac{2D_1+d_1}{2D+d}=\dfrac{d}{d_1}$; but assuming that both d and d_1 are very small compared with D and D_1; we have $\dfrac{D_1}{D}=\dfrac{d}{d_1}$;

Also $\dfrac{R_1}{R}=\dfrac{\frac{D}{D^1}}{\frac{D_1}{D}}=\dfrac{D^2}{D_1^2}$; that is, the resistance at each annulus is inversely as the square of its diameter.

To calculate the total strain,

let P be the fluid pressure per square inch inside,

r the internal radius, t the thickness of ring,

x any variable distance from the internal surface,

S the strain on the internal surface.

Then $S=Pr$; and the strain at x, is $S_x=\dfrac{S.r^2}{(r+x)^2}$.

Whence the total strain on the whole ring of metal

$$={}_0\int^t_x\cdot\frac{Sr^2}{(r+x)^2}=Sr^2\left(\frac{1}{r}-\frac{1}{r+t}\right)=S.\,\frac{rt}{1+t}.$$

To calculate the thickness t,

let C be the cohesive strength of the material per square inch, or similarly to P in any way,

Pr is the strain on the interior surface as before,

and $C\,\dfrac{rt}{r+t}$ is the extreme resistance that the cylinder can exert;

With mere equilibrium, we must have

$$Pr=C.\frac{rt}{r+t};\ \text{so that then } t=\frac{P.r}{C-P};$$

With safety, using a coefficient of one third, as usually adopted with hydraulic presses, then C_1 the safe resistance $=\frac{1}{3}C$;

$$\text{and } t=\frac{Pr}{\frac{1}{3}C-P};=\frac{Pr}{C_1-P}.$$

As this theoretical method fails in general application, it is usual to fall back on empirical rules.

Empirical rules.—One of these is,
$t=0{\cdot}000\,04Hd+0{\cdot}1d^{\frac{1}{2}}+0{\cdot}15$; where t, d, and H the head of water pressure are all in inches.

Another is $t=\frac{1}{5}d^{\frac{1}{2}}$; a formula independent of pressure, or perhaps assuming that d is chosen under some special circumstances that allow of a relation between d and H.

The following tables give some values of thickness used in practice:

Cast Iron Water-pipes; adopted in the Rio de Janeiro Waterworks.

Diameter of Pipe	Thickness	Length without socket	Socket	Weight without ring or socket	Total weight with ring and socket
m. inches	inch.	feet	inches	cwt. qrs lbs.	cwt. qrs. lbs.
0·80 or $31\frac{1}{2}$	$1\frac{3}{16}$	12	$5\frac{14}{16}$	40 2 4	43 3 23
0·80 „ $31\frac{1}{2}$	$1\frac{3}{16}$	9	$5\frac{14}{16}$	30 1 17	33 3 8
0·50 „ $19\frac{11}{16}$	1	12	$5\frac{2}{16}$	21 1 13	22 3 27
0·50 „ $19\frac{11}{16}$	1	9	$5\frac{2}{16}$	16 0 3	17 2 17
0·40 „ $15\frac{12}{16}$	$\frac{14}{16}$	12	$5\frac{2}{16}$	15 0 13	16 1 14
0·40 „ $15\frac{12}{16}$	$\frac{14}{16}$	9	$5\frac{2}{16}$	11 1 10	12 2 11
0·30 „ $11\frac{13}{16}$	$\frac{13}{16}$	9	$4\frac{15}{16}$	7 3 7	8 2 15
0·30 „ $11\frac{13}{16}$	$\frac{11}{16}$	9	$4\frac{15}{16}$	6 3 27	7 2 27
0·25 „ $9\frac{13}{16}$	$\frac{10}{16}$	9	$4\frac{15}{16}$	5 0 23	5 3 5
0·20 „ $7\frac{14}{16}$	$\frac{9}{16}$	9	$4\frac{5}{16}$	3 2 18	4 0 6
0·15 „ $5\frac{14}{16}$	$\frac{8}{16}$	9	$4\frac{5}{16}$	2 2 10	2 3 14
0·10 „ $3\frac{14}{16}$	$\frac{8}{16}$	9	$4\frac{5}{16}$	1 2 16	1 3 10

Testing pressure 15 atmospheres; for $31\frac{1}{2}''$ pipes 20 atmospheres; specific gravity of iron taken at 7·20.

Cast Iron Water-pipes adopted at Glasgow.

Diameter	Thickness	Weight incl. Socket	Working head	Diameter	Thickness	Weight incl. Socket	Working head
"	"	cwts. qrs. lbs.	feet	"	"	cwt. qrs. lbs.	feet
33	1	39 1 25	210	14	$\frac{11}{16}$	8 3 25	290
30	$1\frac{1}{4}$	44 0 3	300	14	$\frac{10}{16}$	8 0 25	250
30	1	35 3 5	230	14	$\frac{9}{16}$	7 2 0	200
24	1	28 1 23	300	12	$\frac{5}{8}$	6 3 13	290
20	$\frac{7}{8}$	16 0 4	270	12	$\frac{9}{16}$	6 0 26	240
20	$\frac{3}{4}$	13 3 25	240	10	$\frac{9}{16}$	5 0 16	300
18	$\frac{13}{16}$	13 1 12	300	9	$\frac{9}{16}$	4 2 24	„
18	$\frac{3}{4}$	12 1 19	260	8	$\frac{1}{2}$	3 2 23	„
18	$\frac{1}{1}$	11 1 27	230	7	$\frac{1}{2}$	3 1 1	„
16	$\frac{3}{4}$	10 3 27	300	6	$\frac{7}{16}$	2 1 27	„
16	$\frac{11}{16}$	10 0 18	250	5	$\frac{13}{32}$	1 3 24	„
16	$\frac{5}{8}$	9 1 9	200	4	$\frac{3}{8}$	1 1 20	„
15	$\frac{11}{16}$	9 2 3	270	3	$\frac{3}{8}$	1 0 10	„
15	$\frac{9}{16}$	7 3 25	180	2	$\frac{3}{8}$	0 2 4	300

Testing strain double the working pressure.

The lengths are 9 feet excluding socket; but for 24" pipes and upwards the length is 12 feet; and for 2" pipe 6 feet.

Absorption and Strength of Cylindrical Stoneware Pipes.

(By Baldwin Latham, C.E.)

Maker and Place	Diam.	Thickness	Length	Weight when dry	Weight after 24 hours' immersion	Percentage of absorption
		"	' "	lbs.	lbs.	
Doulton . .	6"	0·75	1 11	31	31·25	0·806
Doulton, London .		0·72	1 11	29·5	29·75	0·85
Fisher . . .		0·63	2 0	28	28·75	2·68
Wortley . .		0·74	1 11	30·5	31·75	4·10
Doulton, London .	9"	0·87	2 0	57·75	58·75	1·73
Huddersfield .		0·92	2 4	73	73·75	1·03
Cliff, Wortley .		0·81	2 4	60·5	63·25	4·54
Aylesford . .		1·00	2 0	58	62	6·89
Doulton, Stafford .	12"	1·05	2 0	96·0	97·5	1·56
Fisher . . .		1	1 11	84	88	4·76
Stiff . . .		1·02	1 10	66·25	67·5	1·88
Wilcox, Wortley .		1·03	1 11	79·5	82·5	3·77
Doulton, London .	15"	1·06	1 11	116·5	117·0	0·43
Doulton, Stafford .		1·26	2 6	132	139	5·30
Wilcox . . .		1·72	1 10	130	137	5·38
Ingham . .		·31	2 6	165	174·5	5·75
Doulton . .	18"	1·43	2 4	221	226	2·26
" . .		1·38	2 5	210	217	3·33

Maker and Place	Diam.	Thickness	Length	Bursting pressure	Tensile strength	Resistance to crushing
		"	' "	B.	T.	C.
Doulton, Stafford .	6"	0·65	1 11	50	230·7	1742 to 2956
" London .		0·72	1 11	10	41·6	
Ingham, Wortley .		0·48	1 11	4	25	
" "		0·69	1 11	70	304·3	
Doulton, London .	9"	0·84	2 0	40	214·2	2470 to 3561
" Stafford .		0·79	1 11	20	113·9	
Aylesford . .		1·00	2 0	45	202·5	
Cliff, Wortley .		0·84	2 4	60	321·4	
Doulton, Stafford .	12"	1·07	2 0	7	39·2	2834 to 2956
Wilcox, Wortley .		0·94	1 11	7	44·6	
Doulton, London .	15"	1·19	2 5	33	207·9	Not tested
" Stafford .		1·10	1 10	20	136·3	
Ingham, Wortley .		1·15	2 5	20	130·4	
Seacombe, Ruabon		1·10	1 10	63	429·5	

B, T, and C are all in lbs. per sq. inch.

Number 6.—Dam of rectangular section, or impounding wall.

The fluid pressure will act at the centre of pressure, which in this case is at two-thirds the depth. First, to determine where the resultant thrust will intersect the base of the wall,

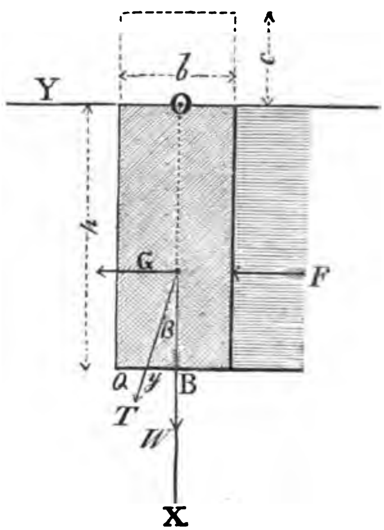


FIGURE 10.

Let F be the fluid pressure,

W the weight of the wall,

h the height of the wall,

b its breadth,

w_1 the weight of 1 cubic foot of the fluid,

w_3 the weight of one cubic foot of the wall.

Also, let O be the origin of rectangular co-ordinates, x vertical, y horizontal co-ordinates,

then $\frac{F}{W}=\frac{w_1 . \frac{1}{2}h^2}{w_3 bh}$; and as $GB=\frac{1}{3}h$; $BQ=y$; and $x=h$;

$y=\frac{1}{6}\frac{w_1 h^3}{w_3 bh}=\frac{1}{6}\frac{w_1 h^2}{w_3 b}$.

Correspondingly, if the wall rise to a height c above water level, the general equation to the line of thrust is

$$y=\frac{1}{6}\frac{w_1(h-c)^3}{w_3 bx} \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{I.}$$

Similarly, also, if a theoretic fluid pressure were used to represent equivalent earth-pressure;

if w were the weight of 1 cubic foot of earth,

$w_2=w.\tan^2(45°-\frac{1}{2}\mu)$; where μ is the angle of repose of the earth;

and w_2 would be the specific gravity of the imaginary fluid that could be applied in either of the two above-mentioned cases, instead of w_1.

As regards stability of position,

let q be the modulus of stability,

qb the fraction of the bottom thickness of wall fixing the distance from its *outer foot* that limits the safe position of the line of thrust,

then, as $x=h$, and $y=b(\frac{1}{2}-q)$,

we have by substitution in the former equation

$$b=\left\{\frac{w_1.(h-c)^3}{w_3.3h(1-2q)}\right\}^{\frac{1}{2}}; \quad . \quad . \quad . \quad . \quad . \quad \text{II.}$$

the breadth of wall here given is consistent with any degree of stability, according to the value given to q; it is generally $\frac{1}{3}$.

Also with theoretic fluid pressure, w_2 would be here also substituted for w_1, to obtain the safe breadth of wall.

Stability as regards sliding of any course of masonry or brickwork.

To prevent sliding; if β be the inclination to verticality made by the resultant thrust, ϕ be the limiting angle of friction, tan β must not exceed tan ϕ;

$$\text{but as } \tan\beta=\frac{w_1}{w_3}.\frac{(x-c)^2}{2bx};$$

hence x must be less than

$$c+\frac{w_1}{w_3}.b\tan\phi\left[1+\frac{w_1}{w_3}.\frac{2c.\operatorname{cotg}\phi}{b}\right] \quad . \quad . \quad . \quad \text{III.}$$

Safety as regards crushing of material.

If W=weight of the wall,

R=resistance to crushing in similar units,

b_1=any augmentation of b necessary to ensure safety,

$$\text{then } b_1=\frac{W}{2R}=\frac{bw_3h}{2R} \quad . \quad . \quad . \quad . \quad . \quad \text{IV.}$$

Number 7.—Dam or impounding wall of trapezoidal section.

If the water-face of the wall be vertical, the centre of fluid-pressure will be at two-thirds the depth of water.

To determine where the resultant thrust will intersect the base of the wall,

Let F be the fluid-pressure;

W the weight of the wall acting in a vertical line through its centre of gravity;

α the inclination to verticality of its back;

a its top width;

c its height above water level.

w_1 the weight of one cubic foot of the fluid;

w_3 the weight of one cubic foot of the wall;

O the origin of rectangular co-ordinates;

g the distance of the centre of gravity from the axis.

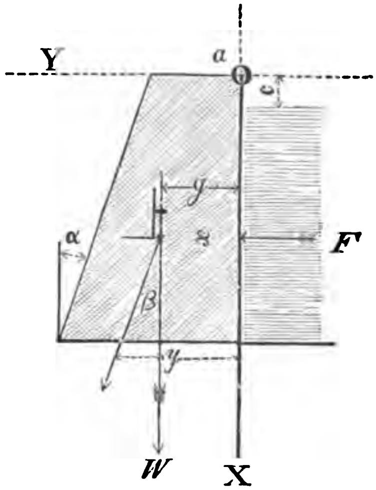


FIGURE 11.

Then we have, by similar triangles, $\dfrac{F}{W}=\dfrac{y-g}{\frac{1}{3}x-c}$;

where $F=\frac{1}{2}w_1(x-c)^2$; and $W=\frac{1}{2}w_3\,x(2a+x\tan\alpha)$;

$$\therefore\ y-g=\frac{w_1(x-c)^3}{w_3\,3x(2a+x\tan\alpha)};\ \text{but}\ g=\frac{\frac{1}{3}x^2\tan^2\alpha+ax\tan\alpha+a^2}{2a+x\tan\alpha};$$

$$\text{hence}\ y=\frac{w_1}{w_3}\cdot\frac{\frac{1}{3}(x-c)^2+\frac{1}{3}x^3\tan^2\alpha+ax^2\tan\alpha+a^2x}{2ax+x^2\tan\alpha}, \quad .\ .\ .\ \text{I.}$$

the general equation to the line of thrust down to any base.

For stability, apply the three principles mentioned in the last solution No. 6 to this case.

1st, a suitable value of q, the modulus.
2nd, that $\tan\beta$ must not exceed $\tan\phi$, the limiting angle of friction.
3rd, apply an augmentation b_1 of base, if required.

The calculations for section of dams being sometimes complicated, a guide for results will be found at page 156 of 'Hydraulic Manual' (fourth edition), where the dimensions of trapezoidal dams up to 40 and 50 feet in height, having either one face vertical or both battering, are tabulated for masonry of three sorts.

Number 8.—*Lofty Dams of Masonry.*

With lofty dams the form of equal strength at every horizontal section will necessarily be a section of curved form, or of double curvature. Presuming that it is more convenient that the sharper curvature shall be at the back (the water-face being the front of the dam), it is desirable to be able to test any section adopted.

To start with, the polygonal trace (of Delocre, No. 25), explained at page 157 of 'Hydraulic Manual' (fourth edition), is a fair assumption, when adapted to the given height, and approximately to the given weight of the masonry to be used.

This approximate section is hence used, for all preliminary purposes of data, for computation in the more exact section.

Without entering at length into the solution of Delocre, it will be noticed that his results take the form of the two

following equations, which control the form and limits of the section; by giving the distances to its edges, from two pressure curves, that can easily be graphically determined from the sectional data, for simple weight of masonry, and for combined weight of masonry and water-pressure; that is, under the hypothesis of stability with water-pressure, and entirely without it.

These two formulæ (which are derived from Bresse, Part 1, No. 32) may be applied at any horizontal section whatever of the dam-section; using the symbols,

l = horizontal thickness of the dam at the section.
W = resultant weight of masonry down to the section.
V = the vertical component of the water-pressure.
H = the horizontal component of the water-pressure.
P = the maximum pressure on any point on the back slope when the water-pressure exists.
q = a distance taken horizontally from the location of the pressure curve at the section to the back slope.
R = the limiting pressure admissible.

The equations are, with and without water-pressure,

$$\text{when } q > \tfrac{1}{3}l; \quad P = 2\left(2 - \frac{3q}{l}\right)\frac{W+V}{l} \overset{=}{\underset{<}{\text{or}}} R;$$

$$\text{when } q < \tfrac{1}{3}l; \quad P = \tfrac{2}{3}\frac{W+V}{q} \overset{=}{\underset{<}{\text{or}}} R.$$

These equations will hold very well when the curvature is slight; but immediately it becomes sharper, a modification of Delocre's results becomes advisable. The resultant, acting at the pressure curve, evidently most affects the point nearest to it on the back slope, and not

the point on a horizontal line with it, as assumed in the above equations. Hence q must then be measured from the pressure curve (at the point of application of the resultant for the horizontal section) in a direction inclined upwards, and normal to the curve of the back slope.

Let α be this variable angle made with the horizon; then the equations become

$$P=2\left(2-\frac{3q}{l}\right)\frac{W+V}{l\cos\alpha};$$

$$P=\tfrac{2}{3}\frac{W+V}{q\cos\alpha};$$

or, they may be expressed in the more convenient form

$$P\cos\alpha=2\left(2-\frac{3q}{l}\right)\frac{W+V}{l}\ \overset{=}{\underset{<}{\text{or}}}\ R;$$

$$P\cos\alpha=\tfrac{2}{3}\frac{W+V}{q}\ \overset{=}{\underset{<}{\text{or}}}\ R.$$

This complication renders the solution for a lofty dam more indeterminate and implicit than before; but the matter cannot be ignored under an increasingly sharp curvature of the back. Without solving the whole case of curvature under abstract conditions the following mode may be adopted:

Having drawn the approximate polygonal section before mentioned, and suited to the case, the pressure curve is laid down for it; the curved back-slope is then laid down by a series of distances q, ignoring the angle α, for about the top quarter-height, where the error would be very small.

When the curvature becomes appreciable, it exists above the horizontal section of reference, and a value of α

may be obtained from the diagram by protraction; similarly in successive lower sections; thus an approximate value of a is always forthcoming, which can be used in the latter equations; and its effect can be applied exactly where it becomes appreciable.

The successive approximation, by working from q to P, and backwards from P to q, is under this method easy and yet sufficiently correct; while the inherent error of the method of Delocre is eliminated.

The result is to add a considerable amount of masonry in the lower part of a lofty dam.

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